

SUPER DUALITY AND CRYSTAL BASES FOR QUANTUM ORTHO-SYMPLECTIC SUPERALGEBRAS II

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ABSTRACT. Let $\mathcal{O}_q^{int}(m|n)$ be a semisimple tensor category of modules over a quantum ortho-symplectic superalgebra of type B, C, D introduced in [16]. It is a natural counterpart of the category of finitely dominated integrable modules over a quantum group of type B, C, D from a viewpoint of super duality. Continuing the previous work on type B and C [16], we classify the irreducible modules in $\mathcal{O}_q^{int}(m|n)$, and prove the existence and uniqueness of their crystal bases in case of type D . A new combinatorial model of classical crystals of type D is introduced, whose super analogue gives a realization of crystals for the highest weight modules in $\mathcal{O}_q^{int}(m|n)$.

1. INTRODUCTION

This is a continuation of our previous work [16] on crystal bases for quantum ortho-symplectic superalgebras. In [16], we constructed a semisimple tensor category $\mathcal{O}_q^{int}(m|n)$ of modules over an quantum superalgebra $U_q(\mathfrak{g}_{m|n})$, where $\mathfrak{g}_{m|n}$ is an ortho-symplectic Lie superalgebra of type B, C, D or $\mathfrak{g} = \mathfrak{b}, \mathfrak{b}^\bullet, \mathfrak{c}, \mathfrak{d}$. This category is characterized by remarkably simple conditions similar to those for the polynomial $U_q(\mathfrak{gl}_{m|n})$ -modules (cf. [2, 3]), while its irreducible modules are q -deformations of infinite-dimensional $\mathfrak{g}_{m|n}$ -modules appearing in a tensor power of a Fock space, which were studied in [4] via Howe duality. Its semisimplicity is based on the fact that $\mathcal{O}_q^{int}(m|n)$ naturally corresponds to the semisimple tensor category $\mathcal{O}_q^{int}(m+n)$ of finitely dominated integrable modules over a quantum enveloping algebra of the corresponding classical Lie (super)algebra \mathfrak{g}_{m+n} from a viewpoint of super duality [6, 7] (more precisely, they are equivalent when $n = \infty$ and $q = 1$).

Motivated by the work on crystal bases of polynomial $U_q(\mathfrak{gl}_{m|n})$ -modules [1], we classified the irreducible modules in $\mathcal{O}_q^{int}(m|n)$ when $\mathfrak{g} = \mathfrak{b}, \mathfrak{b}^\bullet$ and \mathfrak{c} , and then proved the existence and uniqueness of their crystal bases, where the associated crystal is realized in terms of a new combinatorial object called ortho-symplectic tableaux of type B and C , respectively [16].

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In this paper, we establish the same result for $\mathfrak{g} = \mathfrak{d}$ (Theorem 5.7). The strategy for the case $\mathfrak{g} = \mathfrak{d}$ is parallel to that of [16]. But the main ingredient of our proof different from [16] is to formulate the notion of an ortho-symplectic tableau of type D (Definitions 3.1, 3.4, and 3.7), where more technical difficulty enters compared to type B and C .

An ortho-symplectic tableau of type D is a sequence of two-column shaped skew tableaux of type A with certain admissibility conditions on adjacent pairs similar to [16]. Then the crystal of an irreducible highest weight $U_q(\mathfrak{d}_{m|n})$ -module in $\mathcal{O}_q^{int}(m|n)$ is realized as the set of ortho-symplectic tableaux of type D associated to its highest weight, where the underlying tableaux of type A are semistandard tableaux for $\mathfrak{gl}_{m|n} \subset \mathfrak{d}_{m|n}$ [2]. Furthermore, when we replace the underlying tableaux of type A with usual semistandard tableaux for $\mathfrak{gl}_m \subset \mathfrak{d}_m$ (by putting $n = 0$), we obtain a new realization of crystals of integrable highest weight $U_q(\mathfrak{d}_m)$ -modules in $\mathcal{O}_q^{int}(m)$ (type D_m) (Theorems 4.3 and 4.4), which plays a crucial role in this paper.

We remark that the tableaux models for type BCD introduced in this paper and [16] is based on the Fock space model (cf. [4, 8]), while the well-known Kashiwara-Nakashima tableaux (for non-spinor highest weights) [14] are based on the crystals of natural representation and its tensor powers. We expect that our new combinatorial model for classical crystals is of independent interest and can be used for other interesting applications in the future.

Finally, combining with the results in [1] for type A and [16] for type B and C , we conclude that the super duality, when restricted to the integrable modules over the classical Lie algebras, provides a natural semisimple tensor category for Lie superalgebras of types $ABCD$, where a crystal base theory exists. We expect that this can be extended to a more general class of contragredient Lie superalgebras with isotropic simple roots, which includes simple finite-dimensional Lie superalgebras of exceptional types $F(3|1)$, $G(2)$ and $D(2|1, \alpha)$ ($\alpha \in \mathbb{Z}_{>0}$), and whose super duality has been established in [5] recently.

The paper is organized as follows. In Section 2, we briefly recall the notations and results in [16]. In Section 3, we introduce our main combinatorial object called ortho-symplectic tableaux of type D . Then in Section 4, we prove that the set of ortho-symplectic tableaux associated to a given highest weight gives the character of the corresponding irreducible highest weight module over the Lie superalgebra $\mathfrak{d}_{m|n}$. In Section 5, we classify the irreducible $U_q(\mathfrak{d}_{m|n})$ -modules in $\mathcal{O}_q^{int}(m|n)$, and prove the existence and uniqueness of their crystal bases, where the crystals are realized in terms of ortho-symplectic tableaux of type D . In Section 6, we give a proof of Theorem 4.3, which is a main result in Section 4.

2. QUANTUM SUPERALGEBRA $U_q(\mathfrak{d}_{m|n})$ AND THE CATEGORY $\mathcal{O}_q^{int}(m|n)$

2.1. Notations. Throughout this paper, we assume that $m \in \mathbb{Z}_{>0}$ with $m \geq 2$ and $n \in \mathbb{Z}_{>0} \cup \{\infty\}$. Let us recall the following notations for the classical Lie superalgebra $\mathfrak{d}_{m|n}$ of type D in [16, Section 2]:

- $\mathbb{J}_{m|n} = \{ \overline{m} < \dots < \overline{2} < \overline{1} < \frac{1}{2} < \frac{3}{2} < \dots < n - \frac{1}{2} \}$,
- $P_{m|n} = \bigoplus_{a \in \mathbb{J}_{m|n}} \mathbb{Z}\delta_a \oplus \mathbb{Z}\Lambda_{\overline{m}}$: the weight lattice,
- $P_{m|n}^\vee = \bigoplus_{a \in \mathbb{J}_{m|n}} \mathbb{Z}E_a \oplus \mathbb{Z}K'$: the dual weight lattice,
- $I_{m|n} = \{ \overline{m}, \dots, \overline{1}, 0, \frac{1}{2}, \dots, n - \frac{3}{2} \}$,
- $\Pi_{m|n} = \{ \beta_i \mid i \in I_{m|n} \}$: the set of simple roots,
- $\Pi_{m|n}^\vee = \{ \beta_i^\vee \mid i \in I_{m|n} \}$: the set of simple coroots, where

$$\beta_i = \begin{cases} -\delta_{\overline{m}} - \delta_{\overline{m-1}}, & \text{if } i = \overline{m}, \\ \delta_{\overline{k+1}} - \delta_{\overline{k}}, & \text{if } i = \overline{m-1}, \dots, \overline{1}, \\ \delta_{\overline{1}} - \delta_{\frac{1}{2}}, & \text{if } i = 0, \\ \delta_i - \delta_{i+1}, & \text{if } i = \frac{1}{2}, \dots, n - \frac{3}{2}, \end{cases}$$

$$\beta_i^\vee = \begin{cases} -E_{\overline{m}} - E_{\overline{m-1}} + K', & \text{if } i = \overline{m}, \\ E_{\overline{k+1}} - E_{\overline{k}}, & \text{if } i = \overline{m-1}, \dots, \overline{1}, \\ E_{\overline{1}} + E_{\frac{1}{2}}, & \text{if } i = 0, \\ E_i - E_{i+1}, & \text{if } i = \frac{1}{2}, \dots, n - \frac{3}{2}, \end{cases}$$

$$\cdot I_{m|0} = \{ \overline{m}, \dots, \overline{1} \} \text{ and } I_{0|n} = \{ \frac{1}{2}, \dots, n - \frac{3}{2} \}.$$

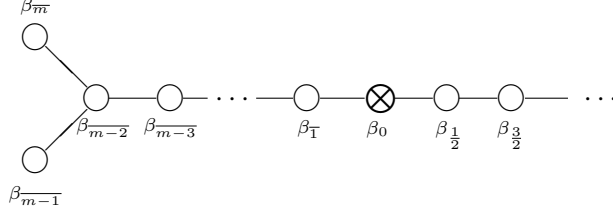
Here, $\mathbb{J}_{m|n}$ is a \mathbb{Z}_2 -graded set with $(\mathbb{J}_{m|n})_0 = \{ \overline{m}, \dots, \overline{1} \}$ and $(\mathbb{J}_{m|n})_1 = \{ 1/2, \dots, n - 1/2 \}$, and we write $|a| = \varepsilon$ for $a \in (\mathbb{J}_{m|n})_\varepsilon$ and $\varepsilon \in \mathbb{Z}_2$. We assume that $\{ \Lambda_{\overline{m}}, \delta_a (a \in \mathbb{J}_{m|n}) \}$ and $\{ K', E_a (a \in \mathbb{J}_{m|n}) \}$ are dual bases with respect to the natural pairing $\langle \cdot, \cdot \rangle$ on $P_{m|n}^\vee \times P_{m|n}$, that is,

$$\langle E_b, \delta_a \rangle = \delta_{ab}, \quad \langle K', \delta_a \rangle = 0, \quad \langle E_a, \Lambda_{\overline{m}} \rangle = 0, \quad \langle K', \Lambda_{\overline{m}} \rangle = 1,$$

for $a, b \in \mathbb{J}_{m|n}$, and $\mathfrak{h}_{m|n}^* := \mathbb{C} \otimes_{\mathbb{Z}} P_{m|n}$ has a symmetric bilinear form $(\cdot | \cdot)$ given by

$$(\lambda | \delta_a) = \langle (-1)^{|a|} E_a - K, \lambda \rangle, \quad (\Lambda_{\overline{m}} | \Lambda_{\overline{m}}) = 0,$$

for $a, b \in \mathbb{J}_{m|n}$ and $\lambda \in \mathfrak{h}_{m|n}^*$. For $i \in I_{m|n}$, let $s_i = 1$ for $i \in \{ \overline{m}, \dots, \overline{1}, 0 \}$, and -1 otherwise. Then $s_j \langle \beta_j^\vee, \mu \rangle = \langle \beta_j, \mu \rangle$ for $j \in I_{m|n}$, $\mu \in \mathfrak{h}_{m|n}^*$. Following [11], the Dynkin diagram associated with the Cartan matrix $A = (a_{ij}) = (\langle \beta_i^\vee, \beta_j \rangle)_{i,j \in I_{m|n}}$ is



For $\Lambda = c\Lambda_{\overline{m}} + \sum_{a \in \mathbb{J}_{m|n}} \lambda_a \delta_a \in P_{m|n}$, we assume that the parity of Λ is $\sum_{a \geq \frac{1}{2}} \lambda_a \pmod{2}$, which we denote by $|\Lambda|$. In particular, we have $|\beta_i| = 0$ for $i \neq 0$ and $|\beta_0| = 1$.

2.2. The quantum superalgebra $U_q(\mathfrak{d}_{m|n})$. Let q be an indeterminate. For $r \geq 0$, put $[r] = \frac{q^r - q^{-r}}{q - q^{-1}}$ and $[r]! = \prod_{k=1}^r [k]$. For $i \in I_{m|n}$, put $q_i = q^{\bar{s}_i}$, where $\bar{s}_i = -s_i$. The quantum superalgebra $U_q(\mathfrak{d}_{m|n})$ is the associative superalgebra (or \mathbb{Z}_2 -graded algebra) with 1 over $\mathbb{Q}(q)$ generated by e_i, f_i ($i \in I_{m|n}$) and q^h ($h \in P_{m|n}^\vee$), which are subject to the following relations [17]:

$$\begin{aligned} \deg(q^h) &= 0, \quad \deg(e_i) = \deg(f_i) = |\beta_i|, \\ q^0 &= 1, \quad q^{h+h'} = q^h q^{h'}, \quad q^h e_i = q^{\langle h, \beta_i \rangle} e_i q^h, \quad q^h f_i = q^{-\langle h, \beta_i \rangle} f_i q^h, \\ e_i f_j - (-1)^{|\beta_i||\beta_j|} f_j e_i &= \delta_{ij} \frac{t_i - t_i^{-1}}{q_i - q_i^{-1}}, \\ z_i z_j - (-1)^{|\beta_i||\beta_j|} z_j z_i &= 0, & \text{if } (\beta_i | \beta_j) = 0, \\ z_i^2 z_j - (q + q^{-1}) z_i z_j z_i + z_j z_i^2 &= 0, & \text{if } i \neq 0 \text{ and } (\beta_i | \beta_j) \neq 0, \\ z_0 z_{\overline{1}} z_0 z_{\frac{1}{2}} + z_{\overline{1}} z_0 z_{\frac{1}{2}} z_0 + z_0 z_{\frac{1}{2}} z_0 z_{\overline{1}} + z_{\frac{1}{2}} z_0 z_{\overline{1}} z_0 - (q + q^{-1}) z_0 z_{\overline{1}} z_{\frac{1}{2}} z_0 &= 0, \end{aligned}$$

for $i, j \in I_{m|n}$, $h, h' \in P_{m|n}^\vee$ and $z = e, f$, where $t_i = q^{\bar{s}_i \beta_i^\vee}$. Recall that there is a Hopf superalgebra structure on $U_q(\mathfrak{d}_{m|n})$, where the comultiplication Δ is given by $\Delta(q^h) = q^h \otimes q^h$, $\Delta(e_i) = e_i \otimes t_i^{-1} + 1 \otimes e_i$, $\Delta(f_i) = f_i \otimes 1 + t_i \otimes f_i$, the antipode S is given by $S(q^h) = q^{-h}$, $S(e_i) = -e_i t_i$, $S(f_i) = -t_i^{-1} f_i$, and the counit ε is given by $\varepsilon(q^h) = 1$, $\varepsilon(e_i) = \varepsilon(f_i) = 0$ for $h \in P_{m|n}^\vee$ and $i \in I_{m|n}$.

2.3. The category $\mathcal{O}_q^{int}(m|n)$. Let \mathcal{P} be the set of partitions and let

$$\mathcal{P}(\mathfrak{d}) = \{ (\lambda, \ell) \in \mathcal{P} \times \mathbb{Z}_{>0} \mid \ell - \lambda_1 - \lambda_2 \in \mathbb{Z}_{\geq 0} \},$$

where $\lambda = (\lambda_1, \lambda_2, \dots)$. For $(\lambda, \ell) \in \mathcal{P}(\mathfrak{d})$, let

$$\Lambda_{m|\infty}(\lambda, \ell) = \ell \Lambda_{\overline{m}} + \lambda_1 \delta_{\overline{m}} + \dots + \lambda_m \delta_{\overline{1}} + \mu_1 \delta_{\frac{1}{2}} + \mu_2 \delta_{\frac{3}{2}} + \dots,$$

where $\mu = (\mu_1, \mu_2, \dots) = (\lambda_{m+1}, \lambda_{m+2}, \dots)'$, the conjugate partition of $(\lambda_{m+1}, \lambda_{m+2}, \dots)$. Put $\mathcal{P}(\mathfrak{d})_{m|n} = \{(\lambda, \ell) \in \mathcal{P}(\mathfrak{d}) \mid \Lambda_{m|\infty}(\lambda, \ell) \in P_{m|n}\}$. For $(\lambda, \ell) \in \mathcal{P}(\mathfrak{d})_{m|n}$, we write $\Lambda_{m|n}(\lambda, \ell) = \Lambda_{m|\infty}(\lambda, \ell)$.

Let $\mathcal{O}_q^{int}(m|n)$ be the category of $U_q(\mathfrak{d}_{m|n})$ -modules M satisfying

- (1) $M = \bigoplus_{\gamma \in P_{m|n}} M_\gamma$ and $\dim M_\gamma < \infty$ for $\gamma \in P_{m|n}$,
- (2) $\text{wt}(M) \subset \bigcup_{i=1}^r \left(\ell_i \Lambda_{\overline{m}} + \sum_{a \in \mathbb{J}_{m|n}} \mathbb{Z}_{\geq 0} \delta_a \right)$ for some $r \geq 1$ and $\ell_i \in \mathbb{Z}_{\geq 0}$,
- (3) $f_{\overline{m}}$ acts locally nilpotently on M ,

where $\text{wt}(M)$ denotes the set of weights of M . For $\Lambda \in P_{m|n}$, let $L_q(\mathfrak{d}_{m|n}, \Lambda)$ denote the irreducible highest weight $U_q(\mathfrak{d}_{m|n})$ -module with highest weight Λ . By [16, Theorems 4.2 and 4.3], we have the following.

Theorem 2.1. $\mathcal{O}_q^{int}(m|n)$ is a semisimple tensor category, and any highest weight module in $\mathcal{O}_q^{int}(m|n)$ is isomorphic to $L_q(\mathfrak{d}_{m|n}, \Lambda_{m|n}(\lambda, \ell))$ for some $(\lambda, \ell) \in \mathcal{P}(\mathfrak{d})_{m|n}$.

2.4. Crystal base. Let M be a $U_q(\mathfrak{d}_{m|n})$ -module in $\mathcal{O}_q^{int}(m|n)$. Let $i \in I_{m|n}$ be given. Suppose that $i \in I_{m|n} \setminus \{0\}$. For $u \in M$ of weight γ , we have a unique expression

$$u = \sum_{k \geq 0, -\langle \beta_i^\vee, \gamma \rangle} f_i^{(k)} u_k,$$

where $e_i u_k = 0$ for all $k \geq 0$. We define the Kashiwara operators \tilde{e}_i and \tilde{f}_i as follows:

$$\begin{aligned} \tilde{e}_i u &= \begin{cases} \sum_k q_i^{l_k - 2k + 1} f_i^{(k-1)} u_k, & \text{if } i \in I_{m|0}, \\ \sum_k f_i^{(k-1)} u_k, & \text{if } i \in I_{0|n}, \end{cases} \\ \tilde{f}_i u &= \begin{cases} \sum_k q_i^{-l_k + 2k + 1} f_i^{(k+1)} u_k, & \text{if } i \in I_{m|0}, \\ \sum_k f_i^{(k+1)} u_k, & \text{if } i \in I_{0|n}, \end{cases} \end{aligned}$$

where $l_k = \langle \beta_i^\vee, \gamma + k\beta_i \rangle$ for $k \geq 0$. If $i = 0$, then we define

$$\tilde{e}_0 u = e_0 u, \quad \tilde{f}_0 u = q_0 f_0 t_0^{-1} u.$$

Let \mathbb{A} denote the subring of $\mathbb{Q}(q)$ consisting of all rational functions which are regular at $q = 0$. We call a pair (L, B) a *crystal base* of M if

- (1) L is an \mathbb{A} -lattice of M , where $L = \bigoplus_{\gamma \in P_{m|n}} L_\gamma$ with $L_\gamma = L \cap M_\gamma$,
- (2) $\tilde{e}_i L \subset L$ and $\tilde{f}_i L \subset L$ for $i \in I_{m|n}$,
- (3) B is a pseudo-basis of L/qL (i.e. $B = B^\bullet \cup (-B^\bullet)$ for a \mathbb{Q} -basis B^\bullet of L/qL),
- (4) $B = \bigsqcup_{\gamma \in P_{m|n}} B_\gamma$ with $B_\gamma = B \cap (L/qL)_\gamma$,
- (5) $\tilde{e}_i B \subset B \sqcup \{0\}$, $\tilde{f}_i B \subset B \sqcup \{0\}$ for $i \in I_{m|n}$,
- (6) for $b, b' \in B$ and $i \in I_{m|n}$, $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$

(see [1]). The set $B/\{\pm 1\}$ has an $I_{m|n}$ -colored oriented graph structure, where $b \xrightarrow{i} b'$ if and only if $\tilde{f}_i b = b'$ for $i \in I_{m|n}$ and $b, b' \in B/\{\pm 1\}$. We call $B/\{\pm 1\}$ the *crystal* of M . For $b \in B$ and $i \in I_{m|n}$, we set $\varepsilon_i(b) = \max\{r \in \mathbb{Z}_{\geq 0} \mid \tilde{e}_i^r b \neq 0\}$ and $\varphi_i(b) = \max\{r \in \mathbb{Z}_{\geq 0} \mid \tilde{f}_i^r b \neq 0\}$. We denote the weight of b by $\text{wt}(b)$.

Let M_i ($i = 1, 2$) be a $U_q(\mathfrak{d}_{m|n})$ -module in $\mathcal{O}_q^{int}(m|n)$ with a crystal base (L_i, B_i) . Then $(L_1 \otimes L_2, B_1 \otimes B_2)$ is a crystal base of $M_1 \otimes M_2$ [1, Proposition 2.8]. The actions of \tilde{e}_i and \tilde{f}_i on $B_1 \otimes B_2$ are as follows.

For $i \in I_{0|n}$, we have

$$(2.1) \quad \begin{aligned} \tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} (\tilde{e}_i b_1) \otimes b_2, & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes (\tilde{e}_i b_2), & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases} \\ \tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} (\tilde{f}_i b_1) \otimes b_2, & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes (\tilde{f}_i b_2), & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases} \end{aligned}$$

For $i \in I_{m|0}$, we have

$$(2.2) \quad \begin{aligned} \tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} b_1 \otimes (\tilde{e}_i b_2), & \text{if } \varphi_i(b_2) \geq \varepsilon_i(b_1), \\ (\tilde{e}_i b_1) \otimes b_2, & \text{if } \varphi_i(b_2) < \varepsilon_i(b_1), \end{cases} \\ \tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} b_1 \otimes (\tilde{f}_i b_2), & \text{if } \varphi_i(b_2) > \varepsilon_i(b_1), \\ (\tilde{f}_i b_1) \otimes b_2, & \text{if } \varphi_i(b_2) \leq \varepsilon_i(b_1). \end{cases} \end{aligned}$$

For $i = 0$, we have

$$(2.3) \quad \begin{aligned} \tilde{e}_0(b_1 \otimes b_2) &= \begin{cases} \pm b_1 \otimes (\tilde{e}_0 b_2), & \text{if } \langle \beta_0^\vee, \text{wt}(b_2) \rangle > 0, \\ (\tilde{e}_0 b_1) \otimes b_2, & \text{if } \langle \beta_0^\vee, \text{wt}(b_2) \rangle = 0, \end{cases} \\ \tilde{f}_0(b_1 \otimes b_2) &= \begin{cases} \pm b_1 \otimes (\tilde{f}_0 b_2), & \text{if } \langle \beta_0^\vee, \text{wt}(b_2) \rangle > 0, \\ (\tilde{f}_0 b_1) \otimes b_2, & \text{if } \langle \beta_0^\vee, \text{wt}(b_2) \rangle = 0. \end{cases} \end{aligned}$$

2.5. Fock space. Let \mathcal{A}_q^+ be an associative $\mathbb{Q}(q)$ -algebra with 1 generated by ψ_a , ψ_a^* , ω_a and ω_a^{-1} for $a \in \mathbb{J}_{m|n}$ subject to the following relations:

$$\begin{aligned} \omega_a \omega_b &= \omega_b \omega_a, \quad \omega_a \omega_a^{-1} = 1, \\ \omega_a \psi_b \omega_a^{-1} &= q^{(-1)^{|a|} \delta_{ab}} \psi_b, \quad \omega_a \psi_b^* \omega_a^{-1} = q^{-(-1)^{|a|} \delta_{ab}} \psi_b^*, \\ \psi_a \psi_b + (-1)^{|a||b|} \psi_b \psi_a &= 0, \quad \psi_a^* \psi_b^* + (-1)^{|a||b|} \psi_b^* \psi_a^* = 0, \\ \psi_a \psi_b^* + (-1)^{|a||b|} \psi_b^* \psi_a &= 0 \quad (a \neq b), \end{aligned}$$

$$\psi_a \psi_a^* = [q\omega_a], \quad \psi_a^* \psi_a = (-1)^{1+|a|}[\omega_a].$$

Here $[q^k \omega_a^{\pm 1}] = \frac{q^k \omega_a^{\pm 1} - q^{-k} \omega_a^{\mp 1}}{q - q^{-1}}$ for $k \in \mathbb{Z}$ and $a \in \mathbb{J}_{m|n}$ (cf. [8]). Let

$$(2.4) \quad \mathcal{V}_q := \mathcal{A}_q^+ |0\rangle$$

be the \mathcal{A}_q^+ -module generated by $|0\rangle$ satisfying $\psi_b^* |0\rangle = 0$ and $\omega_b |0\rangle = q^{-1} |0\rangle$ for $a, b \in \mathbb{J}_{m|n}$. Let \mathbf{B}^+ be the set of sequences $\mathbf{m} = (m_a)$ of non-negative integers indexed by $\mathbb{J}_{m|n}$ such that $m_a \leq 1$ for $|a| = 1$. For $\mathbf{m} = (m_a) \in \mathbf{B}^+$, let

$$\psi_{\mathbf{m}} = \overrightarrow{\prod_{a \in \mathbb{J}_{m|n}} \psi_a^{(m_a)}},$$

where the product is taken in the order on $\mathbb{J}_{m|n}$ and $\psi_a^{(r)} = (\psi_a)^r / [r]!$, $\psi_a^{*(r)} = (\psi_a^*)^r / [r]!$. By similar arguments as in [8, Proposition 2.1], we can check that \mathcal{V}_q is an irreducible \mathcal{A}_q^+ -module with a $\mathbb{Q}(q)$ -linear basis $\{\psi_{\mathbf{m}} |0\rangle \mid \mathbf{m} \in \mathbf{B}^+\}$.

It is shown in [16, Proposition 5.3] that \mathcal{V}_q has a $U_q(\mathfrak{d}_{m|n})$ -module structure, where $\text{wt}(\psi_{\mathbf{m}}) = \Lambda_{\overline{m}} + \sum_{a \in \mathbb{J}_{m|n}} m_a \delta_a$. Since $\mathcal{V}_q \in \mathcal{O}_q^{\text{int}}(m|n)$, $\mathcal{V}_q^{\otimes \ell}$ is completely reducible by Theorem 2.1 for $\ell \geq 1$. Also by [16, Theorem 5.6], \mathcal{V}_q has a crystal base $(\mathcal{L}^+, \mathcal{B}^+)$, where

$$(2.5) \quad \mathcal{L}^+ = \sum_{\mathbf{m} \in \mathbf{B}^+} \mathbb{A} \psi_{\mathbf{m}} |0\rangle, \quad \mathcal{B}^+ = \{ \pm \psi_{\mathbf{m}} |0\rangle \pmod{q\mathcal{L}} \mid \mathbf{m} \in \mathbf{B}^+ \}.$$

3. ORTHO-SYMPLECTIC TABLEAUX OF TYPE D

3.1. Semistandard tableaux. Let us recall some basic terminologies and notations related with tableaux. We refer the reader to [16, Section 3.1]. We assume that \mathcal{A} is a linearly ordered countable set with a \mathbb{Z}_2 -grading $\mathcal{A} = \mathcal{A}_0 \sqcup \mathcal{A}_1$. When \mathcal{A} is (a subset of) $\mathbb{Z}_{>0}$, we assume that $\mathcal{A}_0 = \mathcal{A}$ with the usual linear ordering. For a skew Young diagram λ/μ , we denote by $SST_{\mathcal{A}}(\lambda/\mu)$ be the set of \mathcal{A} -semistandard tableaux of shape λ/μ . For $T \in SST_{\mathcal{A}}(\lambda/\mu)$, $\text{sh}(T)$ denotes the shape of T , $\text{wt}(T) = (m_a)_{a \in \mathcal{A}}$ is the weight of T , where m_a is the number of occurrences of a in T , and $w(T)$ is the word given by reading the entries of T column by column from right to left and from top to bottom in each column.

For $T \in SST_{\mathcal{A}}(\lambda)$ and $a \in \mathcal{A}$, we denote by $a \rightarrow T$ the tableau obtained by the column insertion of a into T (cf. [2, 10]). For a finite word $w = w_1 \dots w_r$ with letters in \mathcal{A} , we define $(w \rightarrow T) = (w_r \rightarrow (\dots (w_1 \rightarrow T)))$. For an \mathcal{A} -semistandard tableau S , we define $(S \rightarrow T) = (w(S) \rightarrow T)$.

For an \mathcal{A} -semistandard tableau S of single-columned shape, we denote by $S(i)$ ($i \geq 1$) the i -th entry from the bottom, and by $\text{ht}(S)$ the height of S .

Let U and V be two \mathcal{A} -semistandard tableaux of single-columned shapes. Let $w = w_1 \dots w_r$ be the word given by rearranging the entries of U and V in weakly decreasing order. Here we assume that for $a \in \mathcal{A}$ occurring in both U and V , we put the letters a in U to the right (resp. left) of those in V if a is even or $a \in \mathcal{A}_0$ (resp. a is odd or $a \in \mathcal{A}_1$). Let $\sigma = (\sigma_1, \dots, \sigma_r)$ be a sequence given by $\sigma_i = +$ if w_i comes from V and $-$ if w_i comes from U . We define

$$(3.1) \quad \sigma(U, V) = (p, q),$$

For $a, b, c \in \mathbb{Z}_{\geq 0}$, let $\lambda(a, b, c) = (2^{b+c}, 1^a)/(1^b)$, which is a skew Young diagram with two columns of heights $a + c$ and $b + c$. For example,

$$\lambda(2, 1, 3) =$$

$$(3.2) \quad T \in SST_{\mathcal{A}}(\lambda(a, b, c)) \text{ if and only if } \sigma(T^{\mathbf{L}}, T^{\mathbf{R}}) = (a - p, b - p)$$

3.3. RSK and signatures. For $\ell \in \mathbb{Z}_{>0}$, let $\mathbf{M}_{\mathcal{A} \times \ell}$ be the set of $\mathcal{A} \times \ell$ matrices $\mathbf{m} = [m_{ai}]$ ($a \in \mathcal{A}$, $i = 1, \dots, \ell$) such that (1) $m_{ai} \in \mathbb{Z}_{\geq 0}$, (2) $m_{ai} \leq 1$ for $|a| = 0$, (3) $\sum_{a,i} m_{ai} < \infty$. Let $\mathbf{m} \in \mathbf{M}_{\mathcal{A} \times \ell}$ be given. For $1 \leq k \leq \ell$, let $\mathbf{m}^{(k)} = [m_{ak}]$ denote the k th column of \mathbf{m} , and $|\mathbf{m}^{(k)}| = \sum_{a \in \mathcal{A}} m_{ak}$. We will often identify each $\mathbf{m}^{(k)}$ with an \mathcal{A} -semistandard tableau of single-columnned shape $(1^{|\mathbf{m}^{(k)}|})$. Similarly, for $a \in \mathcal{A}$, let $\mathbf{m}_{(a)}$ be the a th row of \mathbf{m} , and $|\mathbf{m}_{(a)}| = \sum_{1 \leq k \leq \ell} m_{ak}$. We put $|\mathbf{m}| = \sum_a |\mathbf{m}_{(a)}| = \sum_k |\mathbf{m}^{(k)}|$. We remark that our convention of column indices are increasing from right to left so that $\mathbf{m} = [\mathbf{m}^{(\ell)} : \dots : \mathbf{m}^{(1)}]$.

For $1 \leq k \leq \ell$, let $P(\mathbf{m})^{(k)} = (\mathbf{m}^{(k)} \rightarrow (\cdots (\mathbf{m}^{(2)} \rightarrow \mathbf{m}^{(1)})))$, and let $\lambda^{(k)} = \text{sh}(P(\mathbf{m})^{(k)})$. Put $P(\mathbf{m}) = P(\mathbf{m})^{(\ell)}$ and $\lambda = \lambda^{(\ell)}$. Let $Q(\mathbf{m}) \in SST_{\{1, \dots, \ell\}}(\lambda')$ be such that the subtableau of shape $\lambda^{(k)}/\lambda^{(k-1)}$ is filled with k for $1 \leq k \leq \ell$,

where λ' is the conjugate of λ and $\lambda^{(0)}$ is the empty Young diagram. Then the map $\mathbf{m} \mapsto (P(\mathbf{m}), Q(\mathbf{m}))$ gives a bijection

$$(3.3) \quad \mathbf{M}_{\mathcal{A} \times \ell} \longrightarrow \bigsqcup_{\substack{\lambda \in \mathcal{P} \\ \lambda_1 \leq \ell}} SST_{\mathcal{A}}(\lambda) \times SST_{\{1, \dots, \ell\}}(\lambda'),$$

which is known as the (dual) RSK correspondence.

Suppose that \mathcal{A} has a minimal element, that is, $\mathcal{A} = \{a_1 < a_2 < \dots\}$. For $a \in \mathcal{A}$, we identify $\mathbf{m}_{(a)}$ with a tableau in $SST_{\{1, \dots, \ell\}}(1^p)$ (resp. $SST_{\{1, \dots, \ell\}}(p)$) if $a \in \mathcal{A}_0$ (resp. $a \in \mathcal{A}_1$), where $p = |\mathbf{m}_{(a)}|$. Hence $\mathbf{m}_{(a)}$ can be regarded as an element of a \mathfrak{gl}_{ℓ} -crystal [14] with respect to Kashiwara operators, say \mathcal{E}_i and \mathcal{F}_i for $i = 1, \dots, \ell - 1$, and \mathbf{m} as $\dots \otimes \mathbf{m}_{(a_2)} \otimes \mathbf{m}_{(a_1)}$ following (2.1). Then the map (3.3) is an isomorphism of \mathfrak{gl}_{ℓ} -crystals by [15, Theorems 3.11 and 4.5]. Note that on the righthand side of (3.3), \mathcal{E}_i and \mathcal{F}_i act on $SST_{\{1, \dots, \ell\}}(\lambda')$.

Fix $i \in \{1, \dots, \ell - 1\}$. Let $w = w_1 \dots w_r$ be a finite word with letters in $\{1, \dots, \ell\}$. Let $\sigma = (\sigma_1, \dots, \sigma_r)$ be a sequence with $\sigma_j \in \{+, -, \cdot\}$ such that $\sigma_j = +$ if $w_j = i$, $-$ if $w_j = i + 1$, and \cdot otherwise. We let $\sigma(w; i) = (a, b)$, where a (resp. b) is the number of $-$'s (resp. $+$'s) in $\tilde{\sigma}$ (see Section 3.2). If we regard w as an element of a \mathfrak{gl}_{ℓ} -crystal, then $\mathcal{E}_i w$ (resp. $\mathcal{F}_i w$) is the word replacing $i + 1$ (resp. i) corresponding to the right-most $-$ (resp. the left-most $+$) in $\tilde{\sigma}$ with i (resp. $i + 1$).

Now, let $\sigma(\mathbf{m}; i) = \sigma(w(\mathbf{m}); i)$, where $w(\mathbf{m}) = \dots w(\mathbf{m}_{(a_2)})w(\mathbf{m}_{(a_1)})$ is the concatenation of the words $w(\mathbf{m}_{(a_k)})$ ($1 \leq k \leq \ell$). Also, for $T \in SST_{\{1, \dots, \ell\}}(\lambda)$, let $\sigma(T; i) = \sigma(w(T); i)$. Then the action of \mathcal{E}_i and \mathcal{F}_i on \mathbf{m} and T can be described in terms of $\sigma(\mathbf{m}; i)$ and $\sigma(T; i)$ as in the above paragraph. Since the bijection (3.3) commutes with \mathcal{E}_i and \mathcal{F}_i , we have

$$(3.4) \quad \sigma(\mathbf{m}; i) = \sigma(Q(\mathbf{m}); i).$$

Finally, let U and V be \mathcal{A} -semistandard tableaux of single-columned shapes. Let $\mathbf{m} = [\mathbf{m}^{(2)} : \mathbf{m}^{(1)}] \in \mathbf{M}_{\mathcal{A} \times 2}$, where $\mathbf{m}^{(2)}$ (resp. $\mathbf{m}^{(1)}$) corresponds to U (resp. V). Then it is not difficult to see that

$$(3.5) \quad \sigma(U, V) = \sigma(\mathbf{m}; 1) = (\max\{k \mid \mathcal{E}_1^k \mathbf{m} \neq 0\}, \max\{k \mid \mathcal{F}_1^k \mathbf{m} \neq 0\}).$$

3.4. Ortho-symplectic tableaux of type D .

Definition 3.1.

(1) For $a \in \mathbb{Z}_{\geq 0}$, we define $\mathbf{T}_{\mathcal{A}}^{\mathfrak{d}}(a) = \mathbf{T}_{\mathcal{A}}(a)$ to be the set of $T = (T^{\mathbf{L}}, T^{\mathbf{R}}) \in SST_{\mathcal{A}}(\lambda(a, b, c))$ such that

- (i) $b, c \in 2\mathbb{Z}_{\geq 0}$,
- (ii) $\sigma(T^{\mathbf{L}}, T^{\mathbf{R}}) = (a - r, b - r)$ for some $r = 0, 1$.

We denote r in (ii) by \mathfrak{r}_T . We also define $\overline{\mathbf{T}}_{\mathcal{A}}(0)$ to be set of $T \in SST_{\mathcal{A}}(\lambda(0, b, c+1))$ for some $b, c \in 2\mathbb{Z}_{\geq 0}$.

(2) Let $\mathbf{T}_{\mathcal{A}}^{\text{sp}}$ be the set of \mathcal{A} -semistandard tableaux of single-columned shape. We define $\mathbf{T}_{\mathcal{A}}^{\text{sp}+} = \{T \in \mathbf{T}_{\mathcal{A}}^{\text{sp}} \mid \mathfrak{r}_T = 0\}$ and $\mathbf{T}_{\mathcal{A}}^{\text{sp}-} = \{T \in \mathbf{T}_{\mathcal{A}}^{\text{sp}} \mid \mathfrak{r}_T = 1\}$, where \mathfrak{r}_T is defined to be the residue of $\text{ht}(T)$ modulo 2.

Remark 3.2. Given $T \in SST_{\mathcal{A}}(\lambda(a, b, c))$, one may regard the pair $(T^{\text{L}}, T^{\text{R}})$ as a (not necessarily \mathcal{A} -semistandard) tableau of shape $\lambda(a-k, b-k, c+k)$ sliding T^{R} by k positions down for $0 \leq k \leq \min\{a, b\}$. Then by (3.2), Definition 3.1(ii) means that the pair $(T^{\text{L}}, T^{\text{R}})$ is \mathcal{A} -semistandard of shape $\lambda(a-k, b-k, c+k)$ if and only if k is either 0 or 1, and the maximum of such k is \mathfrak{r}_T .

Example 3.3. Suppose that $\mathcal{A} = \mathbb{J}_{4|\infty}$, and let $T \in \mathbf{T}_{\mathcal{A}}(3)$ be as follows.

$$T = (T^{\text{L}}, T^{\text{R}}) = \begin{array}{c} \boxed{3} \\ \boxed{2} \\ \boxed{\overline{4}} \begin{array}{c} \boxed{\frac{3}{2}} \\ \boxed{\frac{5}{2}} \end{array} \\ \boxed{\overline{1}} \\ \boxed{\frac{1}{2}} \\ \boxed{\frac{3}{2}} \\ \boxed{\frac{3}{2}} \\ \boxed{\frac{3}{2}} \end{array} \in SST_{\mathbb{J}_{4|\infty}}(\lambda(3, 2, 2)) \quad \text{sliding } T^{\text{R}} \text{ down} \quad \begin{array}{c} \boxed{3} \\ \boxed{\overline{4}} \begin{array}{c} \boxed{\frac{3}{2}} \\ \boxed{\frac{5}{2}} \end{array} \\ \boxed{\overline{1}} \\ \boxed{\frac{1}{2}} \\ \boxed{\frac{3}{2}} \\ \boxed{\frac{3}{2}} \\ \boxed{\frac{3}{2}} \end{array} \in SST_{\mathbb{J}_{4|\infty}}(\lambda(2, 1, 3))$$

We have $\mathfrak{r}_T = 1$ since we also have a $\mathbb{J}_{4|\infty}$ -semistandard tableau of shape $\lambda(2, 1, 3)$ after sliding T^{R} down by one position (the tableau on the right).

For $T \in \mathbf{T}_{\mathcal{A}}(a)$, let us identify T with $\mathbf{m} \in \mathbf{M}_{\mathcal{A} \times 2}$ such that T^{L} (resp. T^{R}) correspond to $\mathbf{m}^{(2)}$ (resp. $\mathbf{m}^{(1)}$). Then we define

$$(3.6) \quad ({}^{\text{L}}T, {}^{\text{R}}T) = \mathcal{E}_1^{a-\mathfrak{r}_T} T,$$

that is, the pair of tableaux corresponding to the matrix $\mathcal{E}_1^{a-\mathfrak{r}_T} \mathbf{m}$, and

$$(3.7) \quad (T^{\text{L}*}, T^{\text{R}*}) = \mathcal{F}_1 T,$$

when $\mathfrak{r}_T = 1$. Note that

$$(3.8) \quad \text{ht}({}^{\text{L}}T) = \text{ht}(T^{\text{L}}) - a + \mathfrak{r}_T, \quad \text{ht}({}^{\text{R}}T) = \text{ht}(T^{\text{R}}) + a - \mathfrak{r}_T.$$

One can describe algorithms for $({}^{\text{L}}T, {}^{\text{R}}T)$ and $(T^{\text{L}*}, T^{\text{R}*})$ more explicitly as follows.

Algorithm 1.

- (1) Let \boxed{y} be the box at the bottom of T^{R} .
- (2) Slide down \boxed{y} until the entry x of T^{L} in the same row is no greater (resp. smaller) than y if y is even (resp. odd). If y is even (resp. odd) and no entry of T^{L} is greater than (resp. greater than or equal to) y , we place \boxed{y} to the right of the bottom entry of T^{L} .

- (3) Repeat the process (2) with the entries of T^R above y until there is no moving down of the entries in T^R .
- (4) Move each box \boxed{x} in T^L to the right if its right position is empty. (Indeed the number of such boxes is $a - r_T$.)
- (5) Then ${}^R T$ is the tableau given by the boxes in T^R together with boxes which have come from the left, and ${}^L T$ is the tableau given by the remaining boxes on the left.

In case of Example 3.3, we have

$$T = (T^L, T^R) = \begin{array}{|c|c|} \hline & \boxed{3} \\ \hline & \boxed{2} \\ \hline \boxed{4} & \boxed{\frac{3}{2}} \\ \hline \boxed{1} & \boxed{\frac{5}{2}} \\ \hline \boxed{\frac{1}{2}} & \\ \hline \boxed{\frac{3}{2}} & \\ \hline \boxed{\frac{3}{2}} & \\ \hline \end{array} \rightarrow \begin{array}{|c|c|} \hline & \boxed{3} \\ \hline \boxed{4} & \boxed{2} \\ \hline \boxed{1} & \\ \hline \boxed{\frac{1}{2}} & \boxed{\frac{3}{2}} \\ \hline \boxed{\frac{3}{2}} & \\ \hline \boxed{\frac{3}{2}} & \boxed{\frac{5}{2}} \\ \hline \end{array} \rightarrow \begin{array}{|c|c|} \hline & \boxed{3} \\ \hline \boxed{4} & \boxed{2} \\ \hline & \boxed{1} \\ \hline \boxed{\frac{1}{2}} & \boxed{\frac{3}{2}} \\ \hline & \boxed{\frac{3}{2}} \\ \hline \boxed{\frac{3}{2}} & \boxed{\frac{5}{2}} \\ \hline \end{array} \rightarrow \begin{array}{|c|c|} \hline \boxed{4} & \boxed{3} \\ \hline \boxed{\frac{1}{2}} & \boxed{2} \\ \hline \boxed{\frac{3}{2}} & \boxed{1} \\ \hline & \boxed{\frac{3}{2}} \\ \hline & \boxed{\frac{3}{2}} \\ \hline & \boxed{\frac{5}{2}} \\ \hline \end{array} = ({}^L T, {}^R T)$$

where we arrange $({}^L T, {}^R T)$ so that each of the pairs $({}^L T, T^R)$ and $(T^L, {}^R T)$ shares the same bottom line.

Algorithm 2.

- (1) Let \boxed{x} be the box at the top of T^L .
- (2) Slide upward \boxed{x} until the entry y of T^R in the same row is no smaller than (resp. no greater or equal to) x if x is even (resp. odd). If x is even (resp. odd) and no entry of T^R is smaller than (resp. greater than or equal to) x , we place \boxed{x} to the right of the top entry of T^R .
- (3) Repeat the process (2) with the next entry of T^L below x until there is no moving up of the entries in T^L .
- (4) Choose the lowest box \boxed{y} in T^R whose left position is empty, and then move it to the left. (Since $r_T = 1$, there exists at least one such \boxed{y} .)
- (5) Then T^{L*} is the tableau given by the boxes in T^L together with \boxed{y} , and T^{R*} is the tableau given by the remaining boxes on the right.

In case of Example 3.3, we have

$$T = (T^L, T^R) = \begin{array}{|c|c|} \hline & \boxed{3} \\ \hline & \boxed{2} \\ \hline \boxed{4} & \boxed{\frac{3}{2}} \\ \hline \boxed{1} & \boxed{\frac{5}{2}} \\ \hline \boxed{\frac{1}{2}} & \\ \hline \boxed{\frac{3}{2}} & \\ \hline \boxed{\frac{3}{2}} & \\ \hline \end{array} \rightarrow \begin{array}{|c|c|} \hline \boxed{4} & \boxed{3} \\ \hline & \boxed{2} \\ \hline \boxed{1} & \boxed{\frac{3}{2}} \\ \hline \boxed{\frac{1}{2}} & \boxed{\frac{5}{2}} \\ \hline \boxed{\frac{3}{2}} & \\ \hline \boxed{\frac{3}{2}} & \\ \hline \end{array} \rightarrow \begin{array}{|c|c|} \hline \boxed{4} & \boxed{3} \\ \hline \boxed{2} & \\ \hline \boxed{1} & \boxed{\frac{3}{2}} \\ \hline \boxed{\frac{1}{2}} & \boxed{\frac{5}{2}} \\ \hline \boxed{\frac{3}{2}} & \\ \hline \boxed{\frac{3}{2}} & \\ \hline \end{array} \rightarrow \begin{array}{|c|c|} \hline \boxed{4} & \boxed{3} \\ \hline \boxed{2} & \boxed{\frac{3}{2}} \\ \hline \boxed{1} & \boxed{\frac{5}{2}} \\ \hline \boxed{\frac{1}{2}} & \\ \hline \boxed{\frac{3}{2}} & \\ \hline \boxed{\frac{3}{2}} & \\ \hline \end{array} = (T^{L*}, T^{R*})$$

Note that each of the pairs $(T^{\mathbf{L}*}, T^{\mathbf{L}})$ and $(T^{\mathbf{R}}, T^{\mathbf{R}*})$ shares the same bottom line.

Definition 3.4.

- (1) For $T \in \mathbf{T}_{\mathcal{A}}(a)$ and $S \in \mathbf{T}_{\mathcal{A}}(a') \cup \mathbf{T}_{\mathcal{A}}^{\text{sp}}$ with $a \geq a'$, we write $T \prec S$ if
- (i) $\text{ht}(T^{\mathbf{R}}) \leq \text{ht}(S^{\mathbf{L}}) - a' + 2\mathbf{r}_S\mathbf{r}_T$,
 - (ii) for $i \geq 1$, we have

$$\begin{cases} T^{\mathbf{R}*}(i) \leq {}^{\mathbf{L}}S(i), & \text{if } \mathbf{r}_S = \mathbf{r}_T = 1, \\ T^{\mathbf{R}}(i) \leq {}^{\mathbf{L}}S(i), & \text{otherwise,} \end{cases}$$

- (iii) for $i \geq 1$, we have

$$\begin{cases} {}^{\mathbf{R}}T(i + a - a' + \epsilon) \leq S^{\mathbf{L}*}(i), & \text{if } \mathbf{r}_S = \mathbf{r}_T = 1, \\ {}^{\mathbf{R}}T(i + a - a') \leq S^{\mathbf{L}}(i), & \text{otherwise,} \end{cases}$$

where the equality holds in (ii) and (iii) only if the entries are even, and $\epsilon = 1$ if $S \in \mathbf{T}_{\mathcal{A}}^{\text{sp}-}$ and 0 otherwise. Here we assume that $a' = \mathbf{r}_S$, $S = S^{\mathbf{L}} = {}^{\mathbf{L}}S = S^{\mathbf{L}*}$ when $S \in \mathbf{T}_{\mathcal{A}}^{\text{sp}}$.

- (2) For $T \in \mathbf{T}_{\mathcal{A}}(a)$ and $S \in \overline{\mathbf{T}}_{\mathcal{A}}(0)$, define $T \prec S$ if $T \prec S^{\mathbf{L}}$ in the sense of (1), where $S^{\mathbf{L}} \in \mathbf{T}_{\mathcal{A}}^{\text{sp}-}$.

- (3) For $T \in \overline{\mathbf{T}}_{\mathcal{A}}(0)$ and $S \in \overline{\mathbf{T}}_{\mathcal{A}}(0) \cup \mathbf{T}_{\mathcal{A}}^{\text{sp}-}$, define $T \prec S$ if $(T^{\mathbf{R}}, S^{\mathbf{L}}) \in \overline{\mathbf{T}}_{\mathcal{A}}(0)$.

We say that the pair (T, S) is *admissible* when $T \prec S$.

Example 3.5. Consider the pair (T, S)

$$T = \begin{array}{c} \boxed{\overline{3}} \\ \boxed{\overline{2}} \\ \boxed{\overline{4}} \quad \boxed{\frac{3}{2}} \\ \boxed{\overline{1}} \quad \boxed{\frac{5}{2}} \\ \boxed{\frac{1}{2}} \\ \boxed{\frac{3}{2}} \\ \boxed{\frac{3}{2}} \end{array} = \begin{array}{c} \boxed{\overline{2}} \\ \boxed{\overline{1}} \\ \boxed{\overline{1}} \quad \boxed{\frac{7}{2}} \\ \boxed{\frac{5}{2}} \quad \boxed{\frac{9}{2}} \\ \boxed{\frac{7}{2}} \\ \boxed{\frac{9}{2}} \end{array} = S$$

where T is as in Example 3.3. Note that T and S are arranged so that $T^{\mathbf{R}}$ and $S^{\mathbf{R}}$ share the same bottom line. First, we have $4 = \text{ht}(T^{\mathbf{R}}) \leq \text{ht}(S^{\mathbf{L}}) - 2 + 2\mathbf{r}_T\mathbf{r}_S = 4$, which satisfies Definition 3.4(1)(i). Since

$$({}^{\mathbf{L}}S, {}^{\mathbf{R}}S) = \begin{array}{c} \boxed{\overline{1}} \quad \boxed{\overline{2}} \\ \boxed{\frac{5}{2}} \quad \boxed{\overline{1}} \\ \boxed{\frac{7}{2}} \quad \boxed{\frac{7}{2}} \\ \boxed{\frac{9}{2}} \\ \boxed{\frac{9}{2}} \end{array} \quad (S^{\mathbf{L}*}, S^{\mathbf{R}*}) = \begin{array}{c} \boxed{\overline{2}} \quad \boxed{\overline{1}} \\ \boxed{\overline{1}} \quad \boxed{\frac{7}{2}} \\ \boxed{\frac{5}{2}} \quad \boxed{\frac{9}{2}} \\ \boxed{\frac{7}{2}} \\ \boxed{\frac{9}{2}} \end{array}$$

we have

$$(T^{\mathbf{R}*}, {}^{\mathbf{L}}S) = \begin{array}{|c|c|} \hline \overline{3} & \overline{1} \\ \hline \frac{3}{2} & \frac{5}{2} \\ \hline \frac{5}{2} & \frac{7}{2} \\ \hline \frac{5}{2} & \frac{7}{2} \\ \hline \end{array} \quad ({}^{\mathbf{R}}T, S^{\mathbf{L}*}) = \begin{array}{|c|c|} \hline \overline{3} & \overline{2} \\ \hline \overline{2} & \overline{1} \\ \hline \overline{1} & \frac{5}{2} \\ \hline \frac{3}{2} & \frac{7}{2} \\ \hline \frac{3}{2} & \frac{9}{2} \\ \hline \frac{5}{2} & \frac{7}{2} \\ \hline \end{array}$$

which are $\mathbb{J}_{4|\infty}$ -semistandard, and hence $T \prec S$ by Definition 3.4(1)(ii) and (iii). On the other hand, if we have

$$T = \begin{array}{|c|c|} \hline \overline{3} \\ \hline \overline{2} \\ \hline \overline{4} & \frac{3}{2} \\ \hline \overline{1} & \frac{5}{2} \\ \hline \frac{1}{2} \\ \hline \frac{3}{2} \\ \hline \frac{3}{2} \\ \hline \frac{3}{2} \\ \hline \end{array} = S = \begin{array}{|c|c|} \hline \overline{3} & \overline{2} \\ \hline \overline{2} & \overline{1} \\ \hline \overline{1} & \frac{1}{2} \\ \hline \frac{1}{2} & \frac{3}{2} \\ \hline \frac{3}{2} & \frac{7}{2} \\ \hline \frac{5}{2} & \frac{9}{2} \\ \hline \frac{7}{2} \\ \hline \end{array}$$

with $\mathbf{r}_S = 0$, then

$$({}^{\mathbf{L}}S, {}^{\mathbf{R}}S) = \begin{array}{|c|c|} \hline \overline{3} & \overline{2} \\ \hline \overline{2} & \overline{1} \\ \hline \overline{1} & \frac{1}{2} \\ \hline \frac{1}{2} & \frac{3}{2} \\ \hline \frac{5}{2} & \frac{3}{2} \\ \hline \frac{7}{2} & \frac{7}{2} \\ \hline \frac{9}{2} \\ \hline \end{array} \quad (T^{\mathbf{R}}, {}^{\mathbf{L}}S) = \begin{array}{|c|c|} \hline \overline{3} \\ \hline \overline{2} \\ \hline \overline{3} & \overline{1} \\ \hline \overline{2} & \frac{1}{2} \\ \hline \frac{3}{2} & \frac{5}{2} \\ \hline \frac{5}{2} & \frac{7}{2} \\ \hline \end{array} \quad ({}^{\mathbf{R}}T, S^{\mathbf{L}}) = \begin{array}{|c|c|} \hline \overline{3} \\ \hline \overline{2} \\ \hline \overline{1} \\ \hline \overline{3} & \frac{1}{2} \\ \hline \overline{2} & \frac{3}{2} \\ \hline \overline{1} & \frac{5}{2} \\ \hline \frac{3}{2} & \frac{7}{2} \\ \hline \frac{3}{2} \\ \hline \frac{5}{2} \\ \hline \end{array}$$

Hence we also have $T \prec S$.

Remark 3.6. We can describe equivalent conditions for admissibility in terms of signature σ , which will be useful in the proof of Theorem 3.9. Let (T, S) be as in Definition 3.4(1). The condition (ii) is equivalent to saying that

$$(3.9) \quad (T^{\mathbf{R}}, {}^{\mathbf{L}}S) \text{ or } (T^{\mathbf{R}*}, {}^{\mathbf{L}}S) \in SST_{\mathcal{A}}(\lambda(0, b, c)) \text{ with} \\ (b, c) = (\text{ht}(S^{\mathbf{L}}) - \text{ht}(T^{\mathbf{R}}) - a' + \mathbf{r}_S(\mathbf{r}_T + 1), \text{ht}(T^{\mathbf{R}}) - \mathbf{r}_T \mathbf{r}_S),$$

and hence by (3.2) equivalent to

$$(3.10) \quad \sigma(T^{\mathbf{R}}, {}^{\mathbf{L}}S) \text{ or } \sigma(T^{\mathbf{R}*}, {}^{\mathbf{L}}S) = (0, b).$$

In a similar way, the condition (iii) is equivalent to saying that

$$(3.11) \quad \begin{aligned} &({}^R T, S^L) \text{ or } ({}^R T, S^{L*}) \in SST_{\mathcal{A}}(\lambda(a - a' + \epsilon, b, c - \epsilon)) \text{ with} \\ &(b, c) = (\text{ht}(S^L) - \text{ht}(T^R) - a' + \mathbf{r}_T(\mathbf{r}_S + 1), \text{ht}(T^R) + a' - \mathbf{r}_T), \end{aligned}$$

or equivalent to

$$(3.12) \quad \sigma({}^R T, S^L) \text{ or } \sigma({}^R T, S^{L*}) = (a - a' + \epsilon - p, b - p),$$

for some $p \geq 0$. We remark that the condition (i) is equivalent to $\text{ht}(T^R) \leq \text{ht}(S^L) - a' + \mathbf{r}_T(\mathbf{r}_S + 1)$ or $b \geq 0$ since $\text{ht}(T^L)$ and $\text{ht}(S^L) - a'$ are even integers. We have similar conditions as in (3.10) and (3.12) for the pairs (T, S) in Definition 3.4(2) and (3).

Let $(\lambda, \ell) \in \mathcal{P}(\mathfrak{d})$ be given. Let q_{\pm} and r_{\pm} be non-negative integers such that

$$\begin{cases} \ell - 2\lambda_1 = 2q_+ + r_+, & \text{if } \ell - 2\lambda_1 \geq 0, \\ 2\lambda_1 - \ell = 2q_- + r_-, & \text{if } \ell - 2\lambda_1 \leq 0, \end{cases}$$

where $r_{\pm} = 0, 1$. Let $\bar{\lambda} = (\bar{\lambda}_i)_{i \geq 1} \in \mathcal{P}$ be such that $\bar{\lambda}_1 = \ell - \lambda_1$ and $\bar{\lambda}_i = \lambda_i$ for $i \geq 2$. Let

$$(3.13) \quad \begin{aligned} \nu &= \lambda', \quad \bar{\nu} = (\bar{\lambda})', \\ M_+ &= \lambda_1, \quad M_- = \bar{\lambda}_1 = \ell - \lambda_1, \\ L &= M_{\pm} + q_{\pm}. \end{aligned}$$

Note that $2L + r_{\pm} = \ell$. Put

$$(3.14) \quad \widehat{\mathbf{T}}_{\mathcal{A}}(\lambda, \ell) = \begin{cases} \mathbf{T}_{\mathcal{A}}(\nu_1) \times \cdots \times \mathbf{T}_{\mathcal{A}}(\nu_{M_+}) \times (\mathbf{T}_{\mathcal{A}}(0))^{q_+} \times \left(\mathbf{T}_{\mathcal{A}}^{\text{sp}+}\right)^{r_+}, & \text{if } \ell - 2\lambda_1 \geq 0, \\ \mathbf{T}_{\mathcal{A}}(\bar{\nu}_1) \times \cdots \times \mathbf{T}_{\mathcal{A}}(\bar{\nu}_{M_-}) \times (\bar{\mathbf{T}}_{\mathcal{A}}(0))^{q_-} \times \left(\mathbf{T}_{\mathcal{A}}^{\text{sp}-}\right)^{r_-}, & \text{if } \ell - 2\lambda_1 \leq 0, \end{cases}$$

Here, we assume that $\left(\mathbf{T}_{\mathcal{A}}^{\text{sp}\pm}\right)^{r_{\pm}}$ is empty if $r_{\pm} = 0$.

Now we introduce our main combinatorial object.

Definition 3.7. For $(\lambda, \ell) \in \mathcal{P}(\mathfrak{d})$, we define $\mathbf{T}_{\mathcal{A}}^{\mathfrak{d}}(\lambda, \ell) = \mathbf{T}_{\mathcal{A}}(\lambda, \ell)$ to be the set of $\mathbf{T} = (T_L, \dots, T_1, T_0)$ in $\widehat{\mathbf{T}}_{\mathcal{A}}(\lambda, \ell)$ such that $T_{k+1} \prec T_k$ for $0 \leq k \leq L - 1$. We call $\mathbf{T} \in \mathbf{T}_{\mathcal{A}}(\lambda, \ell)$ an *ortho-symplectic tableau of type D and shape (λ, ℓ)* .

Remark 3.8. Here, we are using a convention slightly different from the cases of type B and C in [16], when we define the notion of admissibility and ortho-symplectic tableaux of type D. But we may still apply Definition 3.4 to $\mathbf{T}_{\mathcal{A}}^{\mathfrak{g}}(a)$ and $\mathbf{T}_{\mathcal{A}}^{\text{sp}}$ for $\mathfrak{g} = \mathfrak{b}, \mathfrak{c}$ and $a \geq 0$ in [16], where all tableaux are of residue 0, and define $\mathbf{T}_{\mathcal{A}}^{\mathfrak{g}}(\lambda, \ell)$ for $(\lambda, \ell) \in \mathcal{P}(\mathfrak{g})$ as in Definition 3.7 with the order of product of $\mathbf{T}_{\mathcal{A}}^{\mathfrak{g}}(a)$ and $\mathbf{T}_{\mathcal{A}}^{\text{sp}}$'s in

[16, Definition 6.10] reversed as in (3.14). Then we can check without difficulty that all the results in [16] can be obtained with this version of ortho-symplectic tableaux of type B and C .

Let $x_{\mathcal{A}} = \{x_a \mid a \in \mathcal{A}\}$ be the set of formal commuting variables indexed by \mathcal{A} . For $\lambda \in \mathcal{P}$, let $s_{\lambda}(x_{\mathcal{A}}) = \sum_T x_{\mathcal{A}}^T$ be the super Schur function corresponding to λ , where the sum is over $T \in SST_{\mathcal{A}}(\lambda)$ and $x_{\mathcal{A}}^T = \prod_a x_a^{m_a}$ with $\text{wt}(T) = (m_a)_{a \in \mathcal{A}}$. For $(\lambda, \ell) \in \mathcal{P}(\mathfrak{d})$, put

$$S_{(\lambda, \ell)}(x_{\mathcal{A}}) = z^{\ell} \sum_{\mathbf{T} \in \mathbf{T}_{\mathcal{A}}(\lambda, \ell)} \prod_{k=0}^L x_{\mathcal{A}}^{T_k},$$

where z is another formal variable. First, we have the following Schur positivity of $S_{(\lambda, \ell)}(x_{\mathcal{A}})$ as in the case of type B and C [16, Theorem 6.12].

Theorem 3.9. *For $(\lambda, \ell) \in \mathcal{P}(\mathfrak{d})$, we have*

$$S_{(\lambda, \ell)}(x_{\mathcal{A}}) = z^{\ell} \sum_{\mu \in \mathcal{P}} K_{\mu(\lambda, \ell)} s_{\mu}(x_{\mathcal{A}}),$$

for some non-negative integers $K_{\mu(\lambda, \ell)}$. Moreover, the coefficients $K_{\mu(\lambda, \ell)}$ do not depend on \mathcal{A} .

Proof. Let L be as in (3.13). Let $\mathbf{T} = (T_L, \dots, T_1, T_0) \in \mathbf{T}_{\mathcal{A}}(\lambda, \ell)$ be given. Let $\mathbf{m} = [\mathbf{m}^{(\ell)} : \dots : \mathbf{m}^{(1)}]$ be the unique matrix in $\mathbf{M}_{\mathcal{A} \times \ell}$, where $\mathbf{m}^{(1)}$ corresponds to T_0 , and $[\mathbf{m}^{(2k+1)} : \mathbf{m}^{(2k)}]$ corresponds to T_k for $1 \leq k \leq L$. We assume that T_0 is empty and $\mathbf{m}^{(1)}$ is trivial when $r_{\pm} = 0$.

Put $Q = Q(\mathbf{m})$, which is of $\{1, \dots, \ell\}$ -semistandard and $\text{wt}(Q) = (m_1, m_2, \dots, m_{\ell})$ with $m_i = |\mathbf{m}^{(i)}|$. For convenience, we put for $1 \leq k \leq M_{\pm}$,

$$(3.15) \quad \begin{aligned} m_k^{\mathbf{L}} &= m_{2q_{\pm}+2k+1}, & m_k^{\mathbf{R}} &= m_{2q_{\pm}+2k}, & r_k &= \mathbf{r}_{T_{q_{\pm}+k}}, \\ a_k &= \begin{cases} \nu_{M_++1-k} & \text{if } \ell - 2\lambda_1 \geq 0, \\ \bar{\nu}_{M_-+1-k} & \text{if } \ell - 2\lambda_1 \leq 0. \end{cases} \end{aligned}$$

First, for $1 \leq k \leq L$, we see from $T_{q_{\pm}+k} \in \mathbf{T}_{\mathcal{A}}(a_k)$, (3.4), and (3.5) that

$$(Q1) \quad m_k^{\mathbf{L}} - a_k, m_k^{\mathbf{R}} \in 2\mathbb{Z}_{\geq 0},$$

$$(Q2) \quad m_k^{\mathbf{L}} - a_k \leq m_k^{\mathbf{R}},$$

$$(Q3) \quad \sigma(Q; 2q_{\pm} + 2k) = (a_k - r_k, m_k^{\mathbf{R}} - m_k^{\mathbf{L}} + a_k - r_k).$$

Put

$$Q^{(k)} = \mathcal{F}_{2q_{\pm}+2k+2}^{r_k r_{k+1}} \mathcal{E}_{2q_{\pm}+2k}^{a_k - r_k} Q, \quad Q^{[k]} = \mathcal{F}_{2q_{\pm}+2k}^{r_k r_{k+1}} \mathcal{E}_{2q_{\pm}+2k+2}^{a_{k+1} - r_{k+1}} Q,$$

for $1 \leq k \leq M_{\pm} - 1$. Since $T_{q_{\pm}+k+1} \prec T_{q_{\pm}+k}$ for $1 \leq k \leq M_{\pm} - 1$, we have by Definition 3.4(1), (3.10), and (3.12) that

- (Q4) $m_{k+1}^R \leq m_k^L - a_k + 2r_k r_{k+1}$,
 (Q5) $\sigma(Q^{(k)}; 2q_{\pm} + 2k) = (0, m_k^L - m_{k+1}^R - a_k + r_k(r_{k+1} + 1))$,
 (Q6) $\sigma(Q^{[k]}; 2q_{\pm} + 2k) = (a_{k+1} - a_k - p_k, m_k^L - m_{k+1}^R - a_k + r_{k+1}(r_k + 1) - p_k)$ for some $p_k \geq 0$.

Next, since $T_k \in \mathbf{T}_{\mathcal{A}}(0)$ or $\overline{\mathbf{T}}_{\mathcal{A}}(0)$ for $1 \leq k \leq q_{\pm}$, $T_0 \in \mathbf{T}_{\mathcal{A}}^{\text{sp}\pm}$, and $T_{k+1} \prec T_k$ for $0 \leq k \leq q_{\pm} - 1$, we have by Definition 3.4(3) that

- (Q7) $m_k \in \mathbb{Z}_{\geq 0}$ for $0 \leq k \leq 2q_+$ and $m_k \in \mathbb{Z}_{>0}$ for $0 \leq k \leq 2q_-$,
 (Q8) $m_{k+1} \leq m_k$ for $0 \leq k \leq 2q_{\pm} - 1$,
 (Q9) $\sigma(Q; k) = (0, m_k - m_{k+1})$ for $0 \leq k \leq 2q_{\pm} - 1$.

Finally, put $Q^{[0]} = \mathcal{E}_{2q_{\pm}+2}^{a_1-r_1} Q$. Since $T_{q_{\pm}+1} \prec T_{q_{\pm}}$, we have by Definition 3.4(1) and (2), (3.10), and (3.12) that

- (Q10) $m_1^R \leq m_{2q_{\pm}+1} - r_0 + 2r_0 r_1$,
 (Q11) $\sigma(Q; 2q_{\pm} + 1) = (0, m_{2q_{\pm}+1} - m_1^R + r_0 r_1)$,
 (Q12) $\sigma(Q^{[0]}; 2q_{\pm} + 1) = (a_1 - p_0, m_{2q_{\pm}+1} - m_1^R - r_0 + r_1(r_0 + 1) - p_0)$ for some $p_0 \geq 0$,

where $r_0 = 1$ if $\ell - 2\lambda_1 < 0$, and 0 otherwise

Conversely, for $\mu \in \mathscr{P}$, let (P, Q) be given where $P \in SST_{\mathcal{A}}(\mu)$ and $Q \in SST_{\{1, \dots, \ell\}}(\mu')$ with $\text{wt}(Q) = (m_1, \dots, m_{\ell})$ satisfying the conditions (Q1)–(Q12) for some r_k ($1 \leq k \leq M_{\pm}$) and p_k ($0 \leq k \leq M_{\pm} - 1$). Note that if such Q exists, then r_k and p_k are uniquely determined by (Q3), (Q6), and (Q12). By (3.3), there exists a unique $\mathbf{m} \in \mathbf{M}_{\mathcal{A} \times \ell}$ such that $(P(\mathbf{m}), Q(\mathbf{m})) = (P, Q)$. Then it follows from (3.2), (3.4), (3.5), and Remark 3.6 that there exists a unique $\mathbf{T} \in \mathbf{T}_{\mathcal{A}}(\lambda, \ell)$ which corresponds to \mathbf{m} . Hence, the map (3.3) induces a weight preserving bijection

$$(3.16) \quad \mathbf{T}_{\mathcal{A}}(\lambda, \ell) \longrightarrow \bigsqcup_{\mu \in \mathscr{P}} SST_{\mathcal{A}}(\mu) \times \mathbf{K}_{\mu(\lambda, \ell)},$$

where $\mathbf{K}_{\mu(\lambda, \ell)}$ is the set of $Q \in SST_{\{1, \dots, \ell\}}(\mu')$ with $\text{wt}(Q) = (m_1, \dots, m_{\ell})$ satisfying (Q1)–(Q12). This implies that $S_{(\lambda, \ell)}(x_{\mathcal{A}}) = z^{\ell} \sum_{\mu \in \mathscr{P}} K_{\mu(\lambda, \ell)} s_{\mu}(x_{\mathcal{A}})$, where $K_{\mu(\lambda, \ell)} = |\mathbf{K}_{\mu(\lambda, \ell)}|$. \square

4. CHARACTER FORMULA OF A HIGHEST WEIGHT MODULE

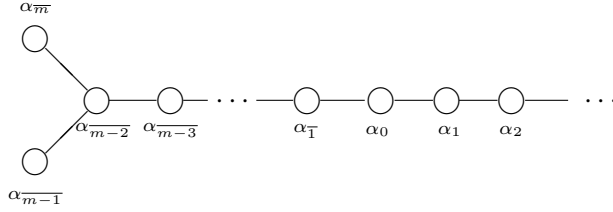
4.1. Lie algebra \mathfrak{d}_{m+n} . We assume the following notations for the classical Lie algebra \mathfrak{d}_{m+n} of type D_{m+n} (see [16] for more details):

- $\mathbb{J}_{m+n} = \{\overline{m} < \dots < \overline{2} < \overline{1} < 1 < 2 < \dots < n\}$,
- $P_{m+n} = \bigoplus_{a \in \mathbb{J}_{m+n}} \mathbb{Z}\delta_a \oplus \mathbb{Z}\Lambda_{\overline{m}}$: the weight lattice,
- $I_{m+n} = \{\overline{m}, \dots, \overline{1}, 0, 1, \dots, n-1\}$,

$\cdot \Pi_{m+n} = \{ \alpha_i \mid i \in I_{m+n} \}$: the set of simple roots, where

$$\alpha_i = \begin{cases} -\delta_{\overline{m}} - \delta_{\overline{m-1}}, & \text{if } i = \overline{m}, \\ \delta_{\overline{k+1}} - \delta_{\overline{k}}, & \text{if } i = \overline{k} \ (\neq \overline{m}), \\ \delta_{\overline{1}} - \delta_1, & \text{if } i = 0, \\ \delta_i - \delta_{i+1}, & \text{if } i = 1, \dots, n-1. \end{cases}$$

Here, we assume that P_{m+n} has a symmetric bilinear form $(\cdot | \cdot)$ such that $(\delta_a | \delta_b) = \delta_{ab}$ and $(\Lambda_{\overline{m}} | \delta_a) = -\frac{1}{2}$ for $a, b \in \mathbb{J}_{m+n}$. The associated Dynkin diagram is



For $(\lambda, \ell) \in \mathcal{P}(\mathfrak{d})$, let

$$\Lambda_{m+\infty}(\lambda, \ell) = \ell \Lambda_{\overline{m}} + \lambda_1 \delta_{\overline{m}} + \dots + \lambda_m \delta_{\overline{1}} + \lambda_{m+1} \delta_1 + \lambda_{m+2} \delta_2 + \dots.$$

Put $\mathcal{P}(\mathfrak{d})_{m+n} = \{ (\lambda, \ell) \in \mathcal{P}(\mathfrak{d}) \mid \Lambda_{m+\infty}(\lambda, \ell) \in P_{m+n} \}$. For $(\lambda, \ell) \in \mathcal{P}(\mathfrak{d})_{m+n}$, we write $\Lambda_{m+n}(\lambda, \ell) = \Lambda_{m+\infty}(\lambda, \ell)$. Then $\{ \Lambda_{m+n}(\lambda, \ell) \mid (\lambda, \ell) \in \mathcal{P}(\mathfrak{d})_{m+n} \}$ is the set of dominant integral weights for \mathfrak{d}_{m+n} . Let Λ_i be the i th fundamental weight for $i \in I_{m+n}$.

4.2. Crystal structure on $\mathbf{T}_{m+n}(\lambda, \ell)$. Put $\mathbf{T}_{m+n}(a) = \mathbf{T}_{\mathbb{J}_{m+n}}(a)$, $\overline{\mathbf{T}}_{m+n}(0) = \overline{\mathbf{T}}_{\mathbb{J}_{m+n}}(0)$, $\mathbf{T}_{m+n}(\lambda, \ell) = \mathbf{T}_{\mathbb{J}_{m+n}}(\lambda, \ell)$, and $\mathbf{T}_{m+n}^{\text{sp}\pm} = \mathbf{T}_{\mathbb{J}_{m+n}}^{\text{sp}\pm}$ for $a \in \mathbb{Z}_{\geq 0}$ and $(\lambda, \ell) \in \mathcal{P}(\mathfrak{d})_{m+n}$.

Let us define an (abstract) \mathfrak{d}_{m+n} -crystal structure on $\mathbf{T}_{m+n}(\lambda, \ell)$. We denote the Kashiwara operators on \mathfrak{d}_{m+n} -crystals by \tilde{e}_i and \tilde{f}_i for $i \in I_{m+n}$, and assume that the tensor product rule follows (2.1).

Recall that \mathbb{J}_{m+n} has a \mathfrak{gl}_{m+n} -crystal structure with respect to \tilde{e}_i and \tilde{f}_i for $i \in I_{m+n} \setminus \{\overline{m}\}$ as follows;

$$\overline{m} \xrightarrow{\overline{m-1}} \overline{m-1} \xrightarrow{\overline{m-2}} \dots \xrightarrow{\overline{1}} \overline{1} \xrightarrow{0} 1 \xrightarrow{1} 2 \xrightarrow{2} \dots$$

where $\text{wt}(a) = \delta_a$ for $a \in \mathbb{J}_{m+n}$. Applying \tilde{e}_i and \tilde{f}_i to the word of a \mathbb{J}_{m+n} -semistandard tableau, we have a \mathfrak{gl}_{m+n} -crystal structure on $SST_{\mathbb{J}_{m+n}}(\lambda/\mu)$ for a skew Young diagram λ/μ [9, 14], where ε_i and φ_i are defined in a usual way. For $\lambda \in \mathcal{P}$, we denote by H_λ the highest weight element in $SST_{\mathbb{J}_{m+n}}(\lambda)$.

Let \mathcal{B} denote one of $\mathbf{T}_{m+n}^{\text{sp}\pm}$, $\overline{\mathbf{T}}_{m+n}(0)$, and $\mathbf{T}_{m+n}(a)$ for $0 \leq a \leq m+n-1$. For $T \in \mathcal{B}$ with $\text{wt}(T) = (m_s)_{s \in \mathbb{J}_{m+n}}$, let

$$\text{wt}(T) = \begin{cases} 2\Lambda_{\overline{m}} + \sum_{s \in \mathbb{J}_{m+n}} m_s \delta_s, & \text{if } \mathcal{B} = \overline{\mathbf{T}}_{m+n}(0) \text{ or } \mathbf{T}_{m+n}(a), \\ \Lambda_{\overline{m}} + \sum_{s \in \mathbb{J}_{m+n}} m_s \delta_s, & \text{if } \mathcal{B} = \mathbf{T}_{m+n}^{\text{sp}\pm}. \end{cases}$$

Since \mathcal{B} is a set of \mathbb{J}_{m+n} -semistandard tableaux, it is a \mathfrak{gl}_{m+n} -crystal with respect to $\tilde{\mathbf{e}}_i$ and $\tilde{\mathbf{f}}_i$ for $i \in I_{m+n} \setminus \{\overline{m}\}$.

Let us define $\tilde{\mathbf{e}}_{\overline{m}}$ and $\tilde{\mathbf{f}}_{\overline{m}}$ on \mathcal{B} . Suppose first that $\mathcal{B} = \mathbf{T}_{m+n}^{\text{sp}\pm}$. For $T \in \mathbf{T}_{m+n}^{\text{sp}}$, let t_1 and t_2 be the first two top entries of T . If $t_1 = \overline{m}$ and $t_2 = \overline{m-1}$, then we define $\tilde{\mathbf{e}}_{\overline{m}}T$ to be the tableau obtained by removing the domino $\begin{smallmatrix} \overline{m} \\ \overline{m-1} \end{smallmatrix}$ from T . Otherwise, we define $\tilde{\mathbf{e}}_{\overline{m}}T = \mathbf{0}$. We define $\tilde{\mathbf{f}}_{\overline{m}}T$ in a similar way by adding a domino $\begin{smallmatrix} \overline{m} \\ \overline{m-1} \end{smallmatrix}$ on top of T . Next, suppose that $\mathcal{B} = \mathbf{T}_{m+n}(a)$ for $0 \leq a \leq m+n$. We regard $\mathbf{T}_{m+n}(a)$ as a subset of $(\mathbf{T}_{m+n}^{\text{sp}})^{\otimes 2}$ by identifying $T = (T^{\text{L}}, T^{\text{R}}) \in \mathbf{T}_{m+n}(a)$ with $T^{\text{R}} \otimes T^{\text{L}}$. Then we apply $\tilde{\mathbf{e}}_{\overline{m}}$ and $\tilde{\mathbf{f}}_{\overline{m}}$ to T following the tensor product rule (2.1). For $T \in \mathcal{B}$, put $\varepsilon_{\overline{m}}(T) = \max\{r \in \mathbb{Z}_{\geq 0} \mid \tilde{\mathbf{e}}_{\overline{m}}^r T \neq \mathbf{0}\}$ and $\varphi_{\overline{m}}(T) = \text{wt}(T) + \varepsilon_{\overline{m}}(T)$.

Lemma 4.1. *Under the above hypothesis, \mathcal{B} is a well-defined \mathfrak{d}_{m+n} -crystal with respect to wt , ε_i , φ_i and $\tilde{\mathbf{e}}_i$, $\tilde{\mathbf{f}}_i$ for $i \in I_{m+n}$.*

Proof. It is clear that $\mathbf{T}_{m+n}^{\text{sp}\pm} \cup \{\mathbf{0}\}$ is invariant under $\tilde{\mathbf{e}}_{\overline{m}}$ and $\tilde{\mathbf{f}}_{\overline{m}}$, and hence becomes a \mathfrak{d}_{m+n} -crystal. So it remains to show that $\mathbf{T}_{m+n}(a) \cup \{\mathbf{0}\}$ or $\overline{\mathbf{T}}_{m+n}(0) \cup \{\mathbf{0}\}$ is invariant under $\tilde{\mathbf{e}}_i$, $\tilde{\mathbf{f}}_i$ for $i \in I_{m+n}$. We will prove the case of $\mathbf{T}_{m+n}(a)$ since the proof for $\overline{\mathbf{T}}_{m+n}(0)$ is similar.

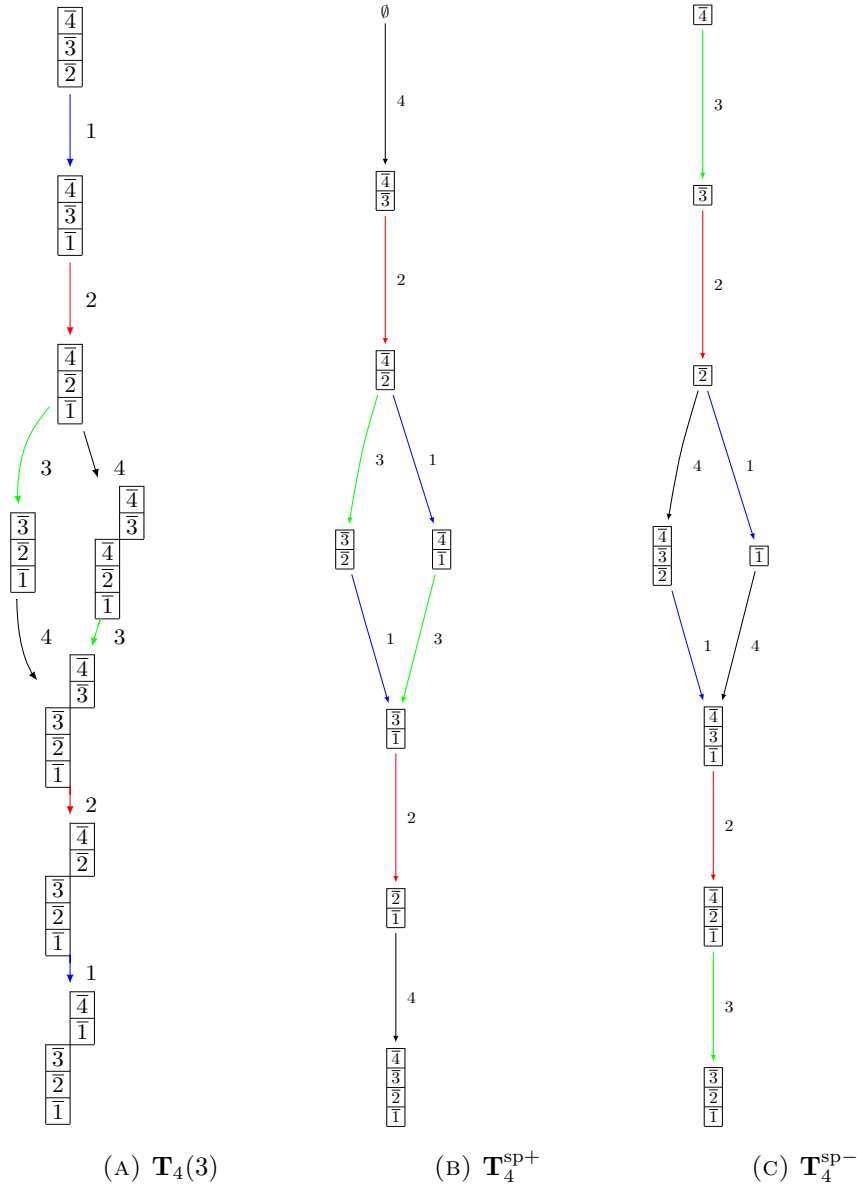
Let $T \in \mathbf{T}_{m+n}(a)$ be given with $\text{sh}(T) = \lambda(a, b, c)$ for some $b, c \in 2\mathbb{Z}_{\geq 0}$. We first observe that $\sigma(T^{\text{L}}, T^{\text{R}})$ is invariant under \tilde{x}_i for $x = \mathbf{e}, \mathbf{f}$ and $i \in I_{m+n} \setminus \{\overline{m}\}$ such that $\tilde{\mathbf{e}}_i T \neq \mathbf{0}$ or $\tilde{\mathbf{f}}_i T \neq \mathbf{0}$, since the map (3.3) is an isomorphism of $(\mathfrak{gl}_{m+n}, \mathfrak{gl}_2)$ -bicrystals.

Next, suppose that $\tilde{\mathbf{e}}_{\overline{m}}T \neq \mathbf{0}$. If $\tilde{\mathbf{e}}_{\overline{m}}T = T^{\text{R}} \otimes (\tilde{\mathbf{e}}_{\overline{m}}T^{\text{L}})$, then $\text{sh}(\tilde{\mathbf{e}}_{\overline{m}}T) = \lambda(a, b+2, c-2)$. Note that the top entry of T^{R} is \overline{m} or $\overline{m-1}$ since otherwise we have $\tilde{\mathbf{e}}_{\overline{m}}T = \mathbf{0}$ by tensor product rule. Then by (3.2) (see also Remark 3.2) we can check without difficulty that

$$\sigma(\tilde{\mathbf{e}}_{\overline{m}}T) = \begin{cases} (a - \mathbf{r}_T, b + 2 - \mathbf{r}_T), & \text{if } \text{ht}(T^{\text{L}}) < \text{ht}(T^{\text{R}}), \\ (a - r, b + 2 - r), & \text{if } \text{ht}(T^{\text{L}}) = \text{ht}(T^{\text{R}}), \end{cases}$$

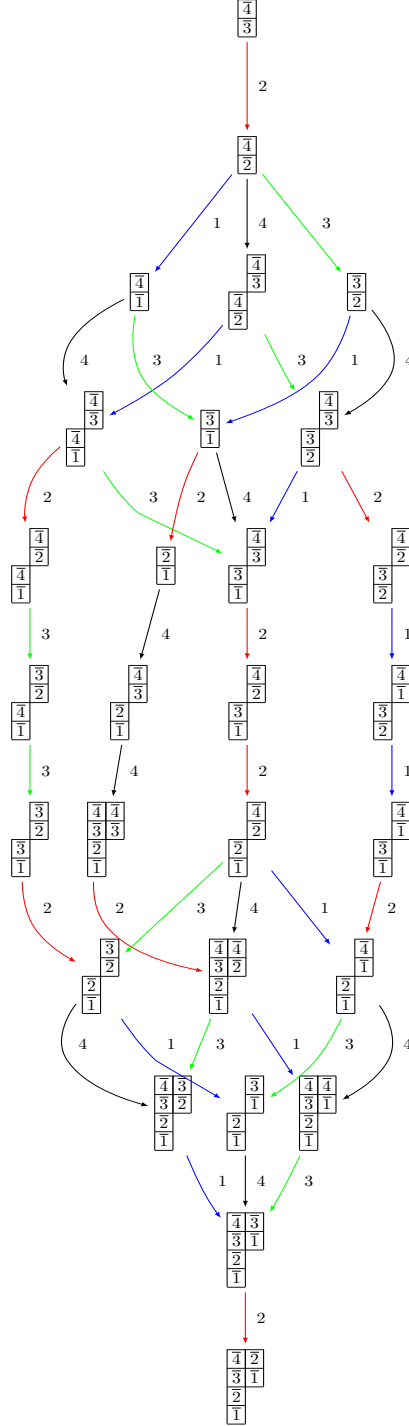
for some $r = 0, 1$. Next, if $\tilde{\mathbf{e}}_{\overline{m}}T = (\tilde{\mathbf{e}}_{\overline{m}}T^{\text{R}}) \otimes T^{\text{L}}$, then $\text{sh}(\tilde{\mathbf{e}}_{\overline{m}}T) = \lambda(a, b-2, c)$ and

$$\sigma(\tilde{\mathbf{e}}_{\overline{m}}T) = \begin{cases} (a - \mathbf{r}_T, b - 2 - \mathbf{r}_T), & \text{if } \text{ht}(T^{\text{L}}) < \text{ht}(T^{\text{R}}) - 2, \\ (a, 0), & \text{if } \text{ht}(T^{\text{L}}) = \text{ht}(T^{\text{R}}) - 2. \end{cases}$$

FIGURE 1. The crystals of type D_4 with $m = 4$ and $n = 0$.

So $\tilde{\mathbf{e}}_m T \in \mathbf{T}_{m+n}(a)$. Hence $\mathbf{T}_{m+n}(a) \cup \{\mathbf{0}\}$ is invariant under $\tilde{\mathbf{e}}_m$. By similar arguments, $\mathbf{T}_{m+n}(a) \cup \{\mathbf{0}\}$ is also invariant under $\tilde{\mathbf{f}}_m$. Therefore, $\mathbf{T}_{m+n}(a)$ is a subcrystal of $(\mathbf{T}_{m+n}^{\text{sp}})^{\otimes 2}$ with respect to wt , ε_i , φ_i and $\tilde{\mathbf{e}}_i$, $\tilde{\mathbf{f}}_i$ for $i \in I_{m+n}$. \square

Let $U_q(\mathfrak{d}_{m+n})$ be the quantized enveloping algebra associated to \mathfrak{d}_{m+n} and let $L_q(\mathfrak{d}_{m+n}, \Lambda)$ be its irreducible highest weight module with highest weight $\Lambda \in P_{m+n}$.

FIGURE 2. The crystal $\mathbf{T}_4(2)$ of type D_4 with $m=4$ and $n=0$.

Recall that $\Lambda_{m+n}((0), 1) = \Lambda_{\overline{m}}$, $\Lambda_{m+n}((1), 1) = \Lambda_{\overline{m-1}}$, and $\Lambda_{m+n}((1^a), 2)$ represents the other fundamental weights for $2 \leq a \leq m+n-1$.

Proposition 4.2.

- (1) $\mathbf{T}_{m+n}^{\text{sp}+}$ is isomorphic to the crystal of $L_q(\mathfrak{d}_{m+n}, \Lambda_{\overline{m}})$.
- (2) $\mathbf{T}_{m+n}^{\text{sp}-}$ is isomorphic to the crystal of $L_q(\mathfrak{d}_{m+n}, \Lambda_{\overline{m-1}})$.
- (3) $\mathbf{T}_{m+n}(a)$ is isomorphic to the crystal of $L_q(\mathfrak{d}_{m+n}, \Lambda_{m+n}((1^a), 2))$ for $0 \leq a \leq m+n-1$.

Proof. (1) Let $T \in \mathbf{T}_{m+n}^{\text{sp}+}$ be given. Let $(\sigma_a)_{a \in \mathbb{J}_{m+n}}$ be sequence of \pm such that $\sigma_a = -$ if and only if a occurs as an entry of T . Then the map sending T to (σ_a) is isomorphism of \mathfrak{d}_{m+n} -crystals from $\mathbf{T}_{m+n}^{\text{sp}+}$ to the crystal of the spin representation $L_q(\mathfrak{d}_{m+n}, \Lambda_{\overline{m}})$ (cf. [14, Section 6.4]). The proof of (2) is almost the same.

(3) We first claim that $\mathbf{T}_{m+n}(a)$ is connected. Let $T \in \mathbf{T}_{m+n}(a)$ be given. We use induction on the number of boxes in $T \in \mathbf{T}_{m+n}(a)$, say $|T|$, to show that T is connected to $H_{(1^a)}$, where $\text{wt}(H_{(1^a)}) = \Lambda_{m+n}((1^a), 2)$. Suppose that $\text{sh}(T) = \mu$. Since $\mathbf{T}_{m+n}(a)$ is a \mathfrak{gl}_{m+n} -crystal, T is connected to H_μ . If $T^{\mathbf{R}}$ is empty, then $\text{ht}(T) = a$ and $T = H_{(1^a)}$. If $T^{\mathbf{R}}$ is not empty, then it has a domino $\begin{smallmatrix} \overline{m} \\ m-1 \end{smallmatrix}$. Hence $\tilde{\mathbf{e}}_{\overline{m}}T \neq \mathbf{0}$ and $|\tilde{\mathbf{e}}_{\overline{m}}T| = |T| - 2$, which completes our induction.

Since $\mathbf{T}_{m+n}^{\text{sp}}$ is a regular crystal and $\mathbf{T}_{m+n}(a) \subset (\mathbf{T}_{m+n}^{\text{sp}})^{\otimes 2}$, $\mathbf{T}_{m+n}(a)$ is also regular, which implies that it is isomorphic to the crystal of $L_q(\mathfrak{d}_{m+n}, \Lambda_{m+n}((1^a), 2))$. \square

Let $(\lambda, \ell) \in \mathcal{P}(\mathfrak{d})_{m+n}$ be given. We keep the notations in (3.13). We regard $\mathbf{T}_{m+n}(\lambda, \ell)$ as a subset of

$$(4.1) \quad \begin{cases} (\mathbf{T}_{m+n}^{\text{sp}+})^{\otimes r_+} \otimes (\mathbf{T}_{m+n}(0))^{\otimes q_+} \otimes \mathbf{T}_{m+n}(\nu_{M_+}) \otimes \cdots \otimes \mathbf{T}_{m+n}(\nu_1), & \text{if } \ell - 2\lambda_1 \geq 0, \\ (\mathbf{T}_{m+n}^{\text{sp}-})^{\otimes r_-} \otimes (\overline{\mathbf{T}}_{m+n}(0))^{\otimes q_-} \otimes \mathbf{T}_{m+n}(\overline{\nu}_{M_-}) \otimes \cdots \otimes \mathbf{T}_{m+n}(\overline{\nu}_1), & \text{if } \ell - 2\lambda_1 \leq 0, \end{cases}$$

by identifying $\mathbf{T} = (T_L, \dots, T_0) \in \mathbf{T}_{m+n}(\lambda, \ell)$ with $T_0 \otimes \cdots \otimes T_L$, and apply $\tilde{\mathbf{e}}_i, \tilde{\mathbf{f}}_i$ on $\mathbf{T}_{m+n}(\lambda, \ell)$ for $i \in I_{m+n}$. We assume that $(\mathbf{T}_{m+n}^{\text{sp}\pm})^{\otimes r_\pm}$ is empty or trivial crystal when $r_\pm = 0$. Then we have the following.

Theorem 4.3. For $(\lambda, \ell) \in \mathcal{P}(\mathfrak{d})_{m+n}$,

- (1) $\mathbf{T}_{m+n}(\lambda, \ell) \cup \{\mathbf{0}\}$ is invariant under $\tilde{\mathbf{e}}_i$ and $\tilde{\mathbf{f}}_i$ for $i \in I_{m+n}$,
- (2) $\mathbf{T}_{m+n}(\lambda, \ell)$ is a connected \mathfrak{d}_{m+n} -crystal with highest weight $\Lambda_{m+n}(\lambda, \ell)$.

Theorem 4.3 immediately implies the following new combinatorial realization of crystal of $L_q(\mathfrak{d}_{m+n}, \Lambda_{m+n}(\lambda, \ell))$, which plays a crucial role in this paper. The proof of Theorem 4.3 is given in Section 6.

Theorem 4.4. *For $(\lambda, \ell) \in \mathcal{P}(\mathfrak{d})_{m+n}$, $\mathbf{T}_{m+n}(\lambda, \ell)$ is isomorphic to the crystal of $L_q(\mathfrak{d}_{m+n}, \Lambda_{m+n}(\lambda, \ell))$.*

Proof. By Proposition 4.2, $\mathbf{T}_{m+n}(a)$, $\overline{\mathbf{T}}_{m+n}(0)$, and $\mathbf{T}_{m+n}^{\text{sp}\pm}$ are regular crystals and so is the crystal (4.1). By Theorem 4.3, $\mathbf{T}_{m+n}(\lambda, \ell)$ is a regular connected crystal with highest weight $\Lambda_{m+n}(\lambda, \ell)$, and hence it is isomorphic to the crystal of $L_q(\mathfrak{d}_{m+n}, \Lambda_{m+n}(\lambda, \ell))$. \square

Let $\mathbb{Z}[P_{m+n}]$ be a group ring of P_{m+n} with a \mathbb{Z} -basis $\{e^\mu \mid \mu \in P_{m+n}\}$. Put $z = e^{\Lambda_{\overline{m}}}$ and $x_a = e^{\delta_a}$ for $a \in \mathbb{J}_{m+n}$. By Theorem 4.4 we have

Corollary 4.5. *For $(\lambda, \ell) \in \mathcal{P}(\mathfrak{d})_{m+n}$, we have*

$$\text{ch}L_q(\mathfrak{d}_{m+n}, \Lambda_{m+n}(\lambda, \ell)) = S_{(\lambda, \ell)}(x_{\mathbb{J}_{m+n}}).$$

4.3. Character of a highest weight module. Now, we have the following combinatorial character formula for the irreducible highest weight module with highest weight $\Lambda_{m|n}(\lambda, \ell)$ for $(\lambda, \ell) \in \mathcal{P}(\mathfrak{d})_{m|n}$, which is the main result in this section.

Theorem 4.6. *For $(\lambda, \ell) \in \mathcal{P}(\mathfrak{d})_{m|n}$, we have*

$$\text{ch}L_q(\mathfrak{d}_{m|n}, \Lambda_{m|n}(\lambda, \ell)) = S_{(\lambda, \ell)}(x_{\mathbb{J}_{m|n}}).$$

That is, the weight generating function of ortho-symplectic tableaux of type D and shape (λ, ℓ) is equal to the character of $L_q(\mathfrak{d}_{m|n}, \Lambda_{m|n}(\lambda, \ell))$.

Proof. By Corollary 4.5 and Theorem 3.9, we have

$$\text{ch}L_q(\mathfrak{d}_{m+\infty}, \Lambda_{m+\infty}(\lambda, \ell)) = S_{(\lambda, \ell)}(x_{\mathbb{J}_{m+\infty}}) = z^\ell \sum_{\mu \in \mathcal{P}} K_{\mu(\lambda, \ell)} s_\mu(x_{\mathbb{J}_{m+\infty}}).$$

Hence by considering the classical limit of $L_q(\mathfrak{d}_{m|\infty}, \Lambda_{m|\infty}(\lambda, \ell))$ (see also [16, Section 4]) and super duality [6, Theorems 4.6 and 5.4], we have

$$\text{ch}L_q(\mathfrak{d}_{m|\infty}, \Lambda_{m|\infty}(\lambda, \ell)) = z^\ell \sum_{\mu \in \mathcal{P}} K_{\mu(\lambda, \ell)} s_\mu(x_{\mathbb{J}_{m|\infty}}) = S_{(\lambda, \ell)}(x_{\mathbb{J}_{m|\infty}}).$$

In particular, $\text{ch}L_q(\mathfrak{d}_{m|n}, \Lambda_{m|n}(\lambda, \ell))$ is obtained by specializing $x_a = 0$ for $a > n+1$, which is equal to $S_{(\lambda, \ell)}(x_{\mathbb{J}_{m|n}})$. \square

5. CRYSTAL BASE OF A HIGHEST WEIGHT MODULE IN $\mathcal{O}_q^{\text{int}}(m|n)$

In this section, we prove that $L_q(\mathfrak{d}_{m|n}, \Lambda_{m|n}(\lambda, \ell))$ is an irreducible module in $\mathcal{O}_q^{\text{int}}(m|n)$ and it has a unique crystal base for $(\lambda, \ell) \in \mathcal{P}(\mathfrak{d})_{m|n}$.

5.1. Crystal structure of $\mathbf{T}_{m|n}(\lambda, \ell)$. Let $U_q(\mathfrak{gl}_{m|n}) = \langle e_i, f_i, q^{\pm E_a} \mid i \in I_{m|n} \setminus \{\bar{m}\}, a \in \mathbb{J}_{m|n} \rangle$ be the subalgebra of $U_q(\mathfrak{d}_{m|n})$ isomorphic to quantized enveloping algebras associated to general linear Lie superalgebras $\mathfrak{gl}_{m|n}$ [17].

We understand $\mathbb{J}_{m|n}$ as the crystal of the natural representation of $U_q(\mathfrak{gl}_{m|n})$, where

$$\bar{m} \xrightarrow{\bar{m}-1} \overline{m-1} \xrightarrow{\bar{m}-2} \cdots \xrightarrow{\bar{1}} \bar{1} \xrightarrow{0} \frac{1}{2} \xrightarrow{\frac{1}{2}} \frac{3}{2} \xrightarrow{\frac{3}{2}} \cdots$$

with $\text{wt}(a) = \delta_a$ for $a \in \mathbb{J}_{m|n}$ [1], and each non-empty word $w = w_1 \cdots w_r$ with letters in $\mathbb{J}_{m|n}$ as $w_1 \otimes \cdots \otimes w_r \in (\mathbb{J}_{m|n})^{\otimes r}$.

Then for a skew Young diagram λ/μ , $SST_{\mathbb{J}_{m|n}}(\lambda/\mu)$ has an (abstract) $\mathfrak{gl}_{m|n}$ -crystal structure [1, Theorem 4.4], where \tilde{e}_i and \tilde{f}_i are defined via the map $SST_{\mathbb{J}_{m|n}}(\lambda/\mu) \rightarrow \bigsqcup_{r \geq 0} (\mathbb{J}_{m|n})^{\otimes r}$ sending T to $w^{\text{rev}}(T)$, the reverse word of $w(T)$. It is known [1, Theorem 5.1] that for $\lambda \in \mathcal{P}$ with $\lambda_{m+1} \leq n$, $SST_{\mathbb{J}_{m|n}}(\lambda)$ is isomorphic to the crystal of an irreducible polynomial $U_q(\mathfrak{gl}_{m|n})$ -module with highest weight $\Lambda_{m|n}(\lambda, 0) \in P_{m|n}$. We denote by H_λ^\natural the highest weight element with weight $\Lambda_{m|n}(\lambda, 0)$, called a *genuine highest weight element* [1, Section 4.2].

Remark 5.1. As in [16], our convention for a crystal base of a $U_q(\mathfrak{gl}_{m|n})$ -module is different from [1]. In our setting, it is a upper crystal base as a $U_q(\mathfrak{gl}_{m|0})$ -module and a lower crystal base as a $U_q(\mathfrak{gl}_{0|n})$ -module (see [16, Remarks 5.1 and 8.1] for more details).

Put $\mathbf{T}_{m|n}^{\text{sp}} = \mathbf{T}_{\mathbb{J}_{m|n}}^{\text{sp}}$ and $\mathbf{T}_{m|n}^{\text{sp}\pm} = \mathbf{T}_{\mathbb{J}_{m|n}}^{\text{sp}\pm}$, which are clearly $\mathfrak{gl}_{m|n}$ -crystals. We also have an $I_{m|n}$ -colored oriented graph structure on $\mathbf{T}_{m|n}^{\text{sp}}$, where $\tilde{e}_{\bar{m}}$ (resp. $\tilde{f}_{\bar{m}}$) is defined by adding (resp. removing) an domino $\begin{smallmatrix} \bar{m} \\ \hline \bar{m}-1 \end{smallmatrix}$ as in the case of $\mathbf{T}_{m+n}^{\text{sp}}$ (see Section 4.2). Then $\mathbf{T}_{m|n}^{\text{sp}\pm} \cup \{\mathbf{0}\}$ is invariant under \tilde{e}_i and \tilde{f}_i for $i \in I_{m|n}$.

Let $\mathbf{m} = (m_a) \in \mathbf{B}^+$ be given (see Section 2.5). Let $T(\mathbf{m}) \in SST_{\mathbb{J}_{m|n}}(1^d)$ be the unique tableaux such that the entries in $T(\mathbf{m})$ are the a 's with $m_a \neq 0$ counting multiplicity as many as m_a times, where $d = \sum_{a \in \mathbb{J}_{m|n}} m_a$. Since \mathbf{B}^+ may be regarded as a crystal of a $U_q(\mathfrak{d}_{m|n})$ -module \mathcal{V}_q by (2.5), we can check the following (see the proof of [16, Theorem 5.6]).

Lemma 5.2. *The map $\Psi^+ : \mathbf{B}^+ \longrightarrow \mathbf{T}_{m|n}^{\text{sp}}$ given by $\Psi^+(\mathbf{m}) = T(\mathbf{m})$ is a bijection which commute with \tilde{e}_i and \tilde{f}_i for $i \in I_{m|n}$. Hence, we may regard $\mathbf{T}_{m|n}^{\text{sp}}$ as a crystal of \mathcal{V}_q , where wt , ε_i and φ_i ($i \in I_{m|n}$) are induced from those on \mathbf{B}^+ via Ψ^+ .*

Next, put $\mathbf{T}_{m|n}(a) = \mathbf{T}_{\mathbb{J}_{m|n}}(a)$, $\overline{\mathbf{T}}_{m|n}(0) = \overline{\mathbf{T}}_{\mathbb{J}_{m|n}}(0)$, and $\mathbf{T}_{m|n}(\lambda, \ell) = \mathbf{T}_{\mathbb{J}_{m|n}}(\lambda, \ell)$ for $a \in \mathbb{Z}_{\geq 0}$ and $(\lambda, \ell) \in \mathcal{P}(\mathfrak{d})_{m|n}$. We regard $\mathbf{T}_{m|n}(a), \overline{\mathbf{T}}_{m|n}(0) \subset (\mathbf{T}_{m|n}^{\text{sp}})^{\otimes 2}$ by identifying T with $T^{\text{L}} \otimes T^{\text{R}}$ (see Remark 5.1).

Lemma 5.3. $\mathbf{T}_{m|n}(a) \cup \{\mathbf{0}\}$ ($a \geq 0$) and $\overline{\mathbf{T}}_{m|n}(0) \cup \{\mathbf{0}\}$ are invariant under \tilde{e}_i and \tilde{f}_i for $i \in I_{m|n}$.

Proof. The proof is almost the same as in Lemma 4.1. \square

Keeping the notations in (3.13), we consider $\mathbf{T}_{m|n}(\lambda, \ell)$ for $(\lambda, \ell) \in \mathcal{P}(\mathfrak{d})_{m|n}$ as a subset of

$$\begin{cases} \mathbf{T}_{m|n}(\nu_1) \otimes \cdots \otimes \mathbf{T}_{m|n}(\nu_{M_+}) \otimes (\mathbf{T}_{m|n}(0))^{q_+} \otimes (\mathbf{T}_{m|n}^{\text{sp}+})^{\otimes r_+}, & \text{if } \ell - 2\lambda_1 \geq 0, \\ \mathbf{T}_{m|n}(\bar{\nu}_1) \otimes \cdots \otimes \mathbf{T}_{m|n}(\bar{\nu}_{M_-}) \otimes (\overline{\mathbf{T}}_{m|n}(0))^{q_-} \otimes (\mathbf{T}_{m|n}^{\text{sp}-})^{\otimes r_-}, & \text{if } \ell - 2\lambda_1 \leq 0, \end{cases}$$

by identifying $\mathbf{T} = (T_L, \dots, T_0) \in \mathbf{T}_{m|n}(\lambda, \ell)$ with $T_L \otimes \cdots \otimes T_0$, and apply \tilde{e}_i and \tilde{f}_i on $\mathbf{T}_{m|n}(\lambda, \ell)$ for $i \in I_{m|n}$. We put

$$(5.1) \quad \mathbf{H}_{(\lambda, \ell)}^{\natural} = H_L \otimes \cdots \otimes H_0,$$

where H_k is empty for $0 \leq k \leq q_+$ when $\ell - 2\lambda_1 \geq 0$, $H_0 = \overline{m}$ and $H_k = \overline{m} \overline{m}$ for $1 \leq k \leq q_-$ when $\ell - 2\lambda_1 \leq 0$, and $H_{q_{\pm}+k} \in SST_{\mathbb{J}_{m|n}}(1^{a_k})$ for $1 \leq k \leq M_{\pm}$ (a_k as in (3.15)) are the unique tableaux such that

$$(H_L \rightarrow (\cdots (H_2 \rightarrow H_0))) = H_{\lambda}^{\natural}.$$

We remark that $H_{q_{\pm}+k}$ is not necessarily equal to $H_{(1^{a_k})}^{\natural}$ in $SST_{\mathbb{J}_{m|n}}(1^{a_k})$ unlike the case of \mathbb{J}_{m+n} -semistandard tableaux (cf. [12, Example 5.8]). Indeed H_{L-k+1} is the k th column of H_{λ}^{\natural} from the left for $k \geq 1$.

Theorem 5.4. For $(\lambda, \ell) \in \mathcal{P}(\mathfrak{d})_{m|n}$,

- (1) $\mathbf{T}_{m|n}(\lambda, \ell) \cup \{\mathbf{0}\}$ is invariant under \tilde{e}_i and \tilde{f}_i for $i \in I_{m|n}$,
- (2) $\mathbf{T}_{m|n}(\lambda, \ell)$ is a connected $I_{m|n}$ -colored oriented graph with a highest weight element $\mathbf{H}_{(\lambda, \ell)}^{\natural}$ of weight $\Lambda_{m|n}(\lambda, \ell)$.

Proof. (1) Since the proof is similar to that of Theorem 4.3 in Section 6, we give a brief sketch of it. In this case, we have $\mathbf{M}_{\mathbb{J}_{m|n} \times 1} = \mathbf{T}_{m|n}^{\text{sp}}$, and hence $\mathbf{M}_{\mathbb{J}_{m+n} \times \ell} = (\mathbf{T}_{m|n}^{\text{sp}})^{\otimes \ell}$ has a $\mathfrak{gl}_{m|n}$ -crystal structure, where we identify $\mathbf{m} = [\mathbf{m}^{(\ell)} : \cdots : \mathbf{m}^{(1)}] \in \mathbf{M}_{\mathbb{J}_{m+n} \times \ell}$ with $\mathbf{m}^{(\ell)} \otimes \cdots \otimes \mathbf{m}^{(1)} \in (\mathbf{T}_{m|n}^{\text{sp}})^{\otimes \ell}$. Then \tilde{e}_i, \tilde{f}_i on $\mathbf{T}_{m|n}(\lambda, \ell)$ coincide with those on $\mathbf{M}_{\mathbb{J}_{m|n} \times \ell}$ for $i \in I_{m|n} \setminus \{\bar{m}\}$ since $\mathbf{T}_{m|n}(\lambda, \ell) \subset (\mathbf{T}_{m|n}^{\text{sp}})^{\otimes \ell}$. Note that $\mathbf{M}_{\mathbb{J}_{m|n} \times \ell}$ is a $(\mathfrak{gl}_{m|n}, \mathfrak{gl}_{\ell})$ -bicrystal and the map (3.3) is an isomorphism of bicrystals [15]. Hence it follows from (3.16) that $\tilde{x}_i \mathbf{T}_{m|n}(\lambda, \ell) \subset \mathbf{T}_{m|n}(\lambda, \ell) \cup \{\mathbf{0}\}$ for $x = e, f$ and $i \in I_{m|n} \setminus \{\bar{m}\}$. The proof for $\tilde{x}_{\bar{m}} \mathbf{T}_{m|n}(\lambda, \ell) \subset \mathbf{T}_{m|n}(\lambda, \ell) \cup \{\mathbf{0}\}$ for $x = e, f$ is the same as in Lemmas 6.2–6.8.

(2) Let $\mathbf{T} = (T_L, \dots, T_0) \in \mathbf{T}_{m|n}(\lambda, \ell)$ be given. As in Lemma 6.9, we use induction on $|\mathbf{T}| = \sum_{k=0}^L |T_k|$ to show that \mathbf{T} is connected to $\mathbf{H}_{(\lambda, \ell)}^{\natural}$. By [1, Theorem

4.8], we may assume that \mathbf{T} is a genuine $\mathfrak{gl}_{m|n}$ -highest weight element, that is, $P := (T_L \rightarrow (\cdots (T_1 \rightarrow T_0))) = H_\mu^\natural$ for some $\mu \in \mathcal{P}$. We will show that $\mathbf{T} = \mathbf{H}_{(\lambda, \ell)}^\natural$ or $\tilde{e}_{\bar{m}}\mathbf{T} \neq \mathbf{0}$, which implies $|\tilde{e}_{\bar{m}}\mathbf{T}| < |\mathbf{T}|$.

Suppose that $\ell - 2\lambda_1 \geq 0$ and consider T_0 or T_1 (if T_0 is empty). From the insertion process for $P := (T_L \rightarrow (\cdots (T_1 \rightarrow T_0)))$, we observe that each k -th entry of T_0 from the top lie in the l -th row of P with $l \leq k$. If T_0 is not empty, then it contains a domino $\begin{smallmatrix} \bar{m} \\ \bar{m}-1 \end{smallmatrix}$ since $P = H_\mu^\natural$. This implies that $\tilde{e}_{\bar{m}}T_0 \neq \mathbf{0}$ and hence $\tilde{e}_{\bar{m}}\mathbf{T} \neq \mathbf{0}$. If T_0 is empty, then T_i is empty for $1 \leq i \leq q_+$ since $T_i \prec T_{i-1}$ for $1 \leq i \leq q_+$. Suppose that $\ell - 2\lambda_1 \leq 0$. By almost the same argument, we conclude that $\tilde{e}_{\bar{m}}\mathbf{T} \neq \mathbf{0}$ or $T_0 = \begin{smallmatrix} \bar{m} \end{smallmatrix}$ and $T_i = \begin{smallmatrix} \bar{m} & \bar{m} \end{smallmatrix}$ for $1 \leq i \leq q_-$.

Now we may assume that T_k is empty for $0 \leq k \leq q_+$ when $\ell - 2\lambda_1 \geq 0$, and $T_0 = \begin{smallmatrix} \bar{m} \end{smallmatrix}$ and $T_k = \begin{smallmatrix} \bar{m} & \bar{m} \end{smallmatrix}$ for $1 \leq k \leq q_-$ when $\ell - 2\lambda_1 \leq 0$.

Consider $T_{q_\pm+1}$. If $\tilde{e}_{\bar{m}}\mathbf{T} = \mathbf{0}$, then by the same argument as in the proof of Lemma 6.9, we have $T_{q_\pm+1}^R$ is empty, and hence $\text{ht}(T_{q_\pm+1}^L) = a_1$. Now we can prove inductively that $T_{q_\pm+k}^R$ is empty and hence $\text{ht}(T_{q_\pm+k}^L) = a_k$ for $1 \leq k \leq M_\pm$. Since $T_{i+1} \prec T_i$ for $0 \leq i \leq L-1$, (T_L, \dots, T_0) itself forms a $\mathbb{J}_{m|n}$ -semistandard tableau H_μ^\natural . Since $|\mathbf{T}| = |\mathbf{H}_{(\lambda, \ell)}^\natural|$ is minimal, we conclude that $\mu = \lambda$ and $\mathbf{T} = \mathbf{H}_{(\lambda, \ell)}^\natural$. The proof completes. \square

5.2. Main result.

Lemma 5.5. *For $a \geq 0$, there exists $\mathbf{v}_a \in \mathcal{V}_q^{\otimes 2}$ such that*

- (1) \mathbf{v}_a is a $U_q(\mathfrak{d}_{m|n})$ -highest weight vector of weight $\Lambda_{m|n}((1^a), 2)$,
- (2) $\mathbf{v}_a \in \mathcal{L}^+ \otimes \mathcal{L}^+$ and $\mathbf{v}_a \equiv \psi_{\mathbf{m}^+(a)}|0\rangle \otimes |0\rangle \pmod{q\mathcal{L}^+ \otimes \mathcal{L}^+}$,

where $\mathbf{m}^+(a) \in \mathbf{B}^+$ is given by $\Psi^+(\mathbf{m}^+(a)) = H_{(1^a)}^\natural$.

Proof. The proof is similar to that of [16, Lemma 5.5]. If $a = 0$, then it is clear that $\mathbf{v}_0 = |0\rangle \otimes |0\rangle$. We assume that $a \geq 1$. Let $\mathbf{M}(a)$ be the set of $\mathbf{m} = [m_{rs}] \in \mathbf{M}_{\mathbb{J}_{m|n} \times 2}$ satisfying the following conditions:

- (1) $m_{r1} + m_{r2} = 1$ for $\bar{m} \leq r \leq \bar{l} + 1$ where $l = \max\{m - a, 0\}$,
- (2) $m_{rs} = 0$ for $r > \frac{1}{2}$ and $s = 1, 2$,
- (3) $m_{\frac{1}{2}1} + m_{\frac{1}{2}2} = \max\{0, a - m\}$,
- (4) $\sum_{r \in \mathbb{J}_{m|n}} m_{r1}$ is even.

Let $\mathbf{m} = [m_{rs}] \in \mathbf{M}(a)$ be given. We write $\mathbf{m} \overset{\bar{m}}{\rightsquigarrow} \mathbf{m}'$ if $m_{\bar{m}2} = m_{\bar{m}-12} = 1$ and \mathbf{m}' is obtained from \mathbf{m} by replacing

$$\begin{bmatrix} m_{\bar{m}2} & m_{\bar{m}1} \\ m_{\bar{m}-12} & m_{\bar{m}-11} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{with} \quad \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

For $i \in \{\overline{m-1}, \dots, \overline{1}\}$, we write $\mathbf{m} \xrightarrow{i} \mathbf{m}'$ if $m_{\overline{i+1}2} = 0$, $m_{\overline{i}2} = 1$ and \mathbf{m}' is obtained from \mathbf{m} by replacing

$$\begin{bmatrix} m_{\overline{i+1}2} & m_{\overline{i}1} \\ m_{\overline{i}2} & m_{\overline{i+1}1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{with} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Similarly, we write $\mathbf{m} \xrightarrow{0} \mathbf{m}'$ if $m_{\overline{1}2} = 0$, $m_{\frac{1}{2}2} \geq 1$ and \mathbf{m}' is obtained from \mathbf{m} by replacing

$$\begin{bmatrix} m_{\overline{1}2} & m_{\overline{1}1} \\ m_{\frac{1}{2}2} & m_{\frac{1}{2}1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ u & v \end{bmatrix} \quad \text{with} \quad \begin{bmatrix} 1 & 0 \\ u-1 & v+1 \end{bmatrix}.$$

Then we have

$$(5.2) \quad e_i(\psi_{\mathbf{m}(2)}|0\rangle \otimes \psi_{\mathbf{m}(1)}|0\rangle) = Q_{\mathbf{m},\mathbf{m}'}(q)e_i(\psi_{\mathbf{m}'(2)}|0\rangle \otimes \psi_{\mathbf{m}'(1)}|0\rangle),$$

for $\mathbf{m} \xrightarrow{i} \mathbf{m}'$, where $Q_{\mathbf{m},\mathbf{m}'}(q)$ is a monomial in q of positive degree given by

$$(5.3) \quad Q_{\mathbf{m},\mathbf{m}'}(q) = \begin{cases} q, & \text{if } i = \overline{m}, \dots, \overline{1}, \\ (-1)^{|\text{wt}(\mathbf{m}^{(2)})|} q^{\langle \beta_0^\vee, \text{wt}(\mathbf{m}^{(1)}) \rangle}, & \text{if } i = 0. \end{cases}$$

Recall $|\text{wt}(\mathbf{m}^{(2)})|$ denotes the degree or parity of $\text{wt}(\mathbf{m}^{(2)})$ (cf. [16, Remark 3.1]).

Let $\mathbf{m}(a) \in \mathbf{M}(a)$ be such that $m_{r1} = 0$ for all $r \in \mathbb{J}_{m|n}$, that is, $\mathbf{m}(a)^{(2)} = \mathbf{m}^+(a)$ and $\mathbf{m}(a)^{(1)}$ is trivial. Then for $\mathbf{m} \in \mathbf{M}(a)$, we have

$$(5.4) \quad \mathbf{m}(a) = \mathbf{m}_0 \xrightarrow{i_1} \mathbf{m}_1 \xrightarrow{i_2} \dots \xrightarrow{i_r} \mathbf{m}_r = \mathbf{m},$$

for some $r \geq 0$, $i_1, \dots, i_r \in \{\overline{m}, \dots, \overline{1}, 0\}$ and $\mathbf{m}_1, \dots, \mathbf{m}_{r-1} \in \mathbf{M}(a)$. Put

$$h(\mathbf{m}) = r, \quad Q_{\mathbf{m}}(q) = \prod_{k=0}^{r-1} Q_{\mathbf{m}_k, \mathbf{m}_{k+1}}(q).$$

Note that $\mathbf{m} \in \mathbf{M}(a)$ is completely determined by its second column $\mathbf{m}^{(2)}$, and under this identification the $\{\overline{m}, \dots, \overline{1}, 0\}$ -colored graph structure on $\mathbf{M}(a)$ with respect to \xrightarrow{i} coincides with the $\mathfrak{d}_{m|1}$ -crystal structure on $\mathbf{T}_{m|1}^{\text{sp}+}$ (see Section 5.1). This implies as in [16, Lemma 8.6] that $h(\mathbf{m})$ and $Q_{\mathbf{m}}(q)$ are independent of a path (5.4) from $\mathbf{m}(a)$ to \mathbf{m} . Put

$$\mathbf{v}_a = \sum_{\mathbf{m} \in \mathbf{M}(a)} (-1)^{h(\mathbf{m})} Q_{\mathbf{m}}(q) \psi_{\mathbf{m}(2)}|0\rangle \otimes \psi_{\mathbf{m}(1)}|0\rangle.$$

Then $\mathbf{v}_a \in \mathcal{L}^+ \otimes \mathcal{L}^+$ and $\mathbf{v}_a \equiv \psi_{\mathbf{m}^+(a)}|0\rangle \otimes |0\rangle \pmod{q\mathcal{L}^+ \otimes \mathcal{L}^+}$.

Consider the pairs $(\mathbf{m}, \mathbf{m}')$ for $\mathbf{m}, \mathbf{m}' \in \mathbf{M}(a)$ with $\mathbf{m} \xrightarrow{i} \mathbf{m}'$ for some $i \in I_{m|n}$. We see that any $\mathbf{m} \in \mathbf{M}(a)$ with $e_i(\psi_{\mathbf{m}(2)}|0\rangle \otimes \psi_{\mathbf{m}(1)}|0\rangle) \neq 0$ belongs to one of these

pairs. Since $h(\mathbf{m}') = h(\mathbf{m}) + 1$ and $Q_{\mathbf{m}'}(q) = Q_{\mathbf{m}}(q)Q_{\mathbf{m},\mathbf{m}'}(q)$, we have by (5.2)

$$\begin{aligned} & e_i \left\{ (-1)^{h(\mathbf{m})} Q_{\mathbf{m}}(q) \psi_{\mathbf{m}(2)}|0\rangle \otimes \psi_{\mathbf{m}(1)}|0\rangle + (-1)^{h(\mathbf{m}')} Q_{\mathbf{m}'}(q) \psi_{\mathbf{m}'(2)}|0\rangle \otimes \psi_{\mathbf{m}'(1)}|0\rangle \right\} \\ &= (-1)^{h(\mathbf{m})} Q_{\mathbf{m}}(q) \left\{ e_i(\psi_{\mathbf{m}(2)}|0\rangle \otimes \psi_{\mathbf{m}(1)}|0\rangle) - Q_{\mathbf{m},\mathbf{m}'}(q) e_i(\psi_{\mathbf{m}'(2)}|0\rangle \otimes \psi_{\mathbf{m}'(1)}|0\rangle) \right\} \\ &= 0. \end{aligned}$$

This implies that $e_i \mathbf{v}_a = 0$ for all $i \in I_{m|n}$, and hence it is a $U_q(\mathfrak{d}_{m|n})$ -highest weight vector. \square

Proposition 5.6. *For $a \geq 0$, let \mathbf{v}_a be as in Lemma 5.5. Then $U_q(\mathfrak{d}_{m|n})\mathbf{v}_a$ is isomorphic to $L_q(\mathfrak{d}_{m|n}, \Lambda_{m|n}((1^a), 2))$, and it has a crystal base $(\mathcal{L}(a), \mathcal{B}(a))$, where*

$$\begin{aligned} \mathcal{L}(a) &= \sum \mathbb{A} \tilde{x}_{i_1} \cdots \tilde{x}_{i_r} \mathbf{v}_a, \\ \mathcal{B}(a) &= \{ \pm \tilde{x}_{i_1} \cdots \tilde{x}_{i_r} \mathbf{v}_a \pmod{q\mathcal{L}(a)} \} \setminus \{0\}, \end{aligned}$$

with $r \geq 0$, $i_1, \dots, i_r \in I_{m|n}$, and $x = e, f$ for each i_k . The crystal $\mathcal{B}(a)/\{\pm 1\}$ is isomorphic to $\mathbf{T}_{m|n}(a)$.

Proof. By Lemma 5.5 and Theorem 2.1, we have

$$U_q(\mathfrak{d}_{m|n})\mathbf{v}_a \cong L_q(\mathfrak{d}_{m|n}, \Lambda_{m|n}((1^a), 2)).$$

Also, it follows from the same argument in [16, Proposition 8.7] that $(\mathcal{L}(a), \mathcal{B}(a))$ is a crystal base of $L_q(\mathfrak{d}_{m|n}, \Lambda_{m|n}((1^a), 2))$, and $\mathcal{B}(a)/\{\pm 1\}$ is isomorphic to $\mathbf{T}_{m|n}(a)$. \square

Now we are ready to state and prove our main theorem in this paper.

Theorem 5.7. *For $(\lambda, \ell) \in \mathcal{P}(\mathfrak{d})_{m|n}$, $L_q(\mathfrak{d}_{m|n}, \Lambda_{m|n}(\lambda, \ell))$ is an irreducible $U_q(\mathfrak{d}_{m|n})$ -module in $\mathcal{O}_q^{\text{int}}(m|n)$, and it has a unique crystal base up to scalar multiplication, whose crystal is isomorphic to $\mathbf{T}_{m|n}(\lambda, \ell)$.*

Proof. Let $(\lambda, \ell) \in \mathcal{P}(\mathfrak{d})_{m|n}$ be given with L as in (3.13). Let $V_{(\lambda, \ell)} = V_L \otimes \cdots \otimes V_0$ be a $U_q(\mathfrak{gl}_{m|n})$ -module, where

$$(5.5) \quad V_i = \begin{cases} U_q(\mathfrak{gl}_{m|n})\mathbf{v}_{\mu_{L-i+1}}, & \text{if } \ell - 2\lambda_1 \geq 0 \text{ and } q_+ + 1 \leq i \leq L, \\ U_q(\mathfrak{gl}_{m|n})|0\rangle \otimes |0\rangle, & \text{if } \ell - 2\lambda_1 \geq 0 \text{ and } 1 \leq i \leq q_+, \\ U_q(\mathfrak{gl}_{m|n})|0\rangle, & \text{if } \ell - 2\lambda_1 \geq 0 \text{ and } i = 0, \\ U_q(\mathfrak{gl}_{m|n})\mathbf{v}_{\bar{\mu}_{L-i+1}}, & \text{if } \ell - 2\lambda_1 < 0 \text{ and } q_- + 1 \leq i \leq L, \\ U_q(\mathfrak{gl}_{m|n})\psi_{\bar{m}}|0\rangle \otimes \psi_{\bar{m}}|0\rangle, & \text{if } \ell - 2\lambda_1 < 0 \text{ and } 1 \leq i \leq q_-, \\ U_q(\mathfrak{gl}_{m|n})\psi_{\bar{m}}|0\rangle, & \text{if } \ell - 2\lambda_1 < 0 \text{ and } i = 0. \end{cases}$$

Here we assume that V_0 is trivial when r_+ or r_- is 0. Then V_i is isomorphic to an irreducible polynomial $U_q(\mathfrak{gl}_{m|n})$ -module, and $V_{(\lambda, \ell)}$ is a completely reducible $U_q(\mathfrak{gl}_{m|n})$ -module with a crystal base [1]. Also, by [16, Theorem 5.6] and Proposition 5.6, we may assume that the crystal lattice of $V_{(\lambda, \ell)}$ is contained in a tensor product of $\mathcal{L}(a)$'s and \mathcal{L}^+ 's, say \mathcal{L} .

The rest of the proof is the same as in [16, Theorem 8.8], which we refer the reader to for more details. First, from the decomposition of $V_{(\lambda, \ell)}$ (cf. [12]), we can find a unique $U_q(\mathfrak{gl}_{m|n})$ -highest weight vector $\mathbf{v}_{(\lambda, \ell)}$ in $V_{(\lambda, \ell)}$ (up to scalar multiplication) such that $U_q(\mathfrak{gl}_{m|n})\mathbf{v}_{(\lambda, \ell)}$ is isomorphic to the irreducible $U_q(\mathfrak{gl}_{m|n})$ -module with highest weight $\Lambda_{m|n}(\lambda, \ell)$ and $\mathbf{v}_{(\lambda, \ell)} \not\equiv 0 \pmod{q\mathcal{L}}$. Since $e_{\bar{m}}V_i = 0$ for all i by construction, $\mathbf{v}_{(\lambda, \ell)}$ is a $U_q(\mathfrak{d}_{m|n})$ -highest weight vector and $U_q(\mathfrak{d}_{m|n})\mathbf{v}_{(\lambda, \ell)} \cong L_q(\mathfrak{d}_{m|n}, \Lambda_{m|n}(\lambda, \ell))$ with $\mathbf{v}_{(\lambda, \ell)} \equiv \pm \mathbf{H}_{(\lambda, \ell)}^\natural \pmod{q\mathcal{L}}$ (see (5.1)). Next if we put

$$\begin{aligned} \mathcal{L}(\lambda, \ell) &= \sum \mathbb{A} \tilde{x}_{i_1} \cdots \tilde{x}_{i_r} \mathbf{v}_{(\lambda, \ell)} \subset \mathcal{L}, \\ \mathcal{B}(\lambda, \ell) &= \{ \pm \tilde{x}_{i_1} \cdots \tilde{x}_{i_r} \mathbf{v}_{(\lambda, \ell)} \pmod{q\mathcal{L}(\lambda, \ell)} \} \setminus \{0\}, \end{aligned}$$

where $r \geq 0$, $i_1, \dots, i_r \in I_{m|n}$, and $x = e, f$ for each i_k , then it follows from Lemma 5.2, Proposition 5.6, and Theorems 4.6 and 5.4 that $(\mathcal{L}(\lambda, \ell), \mathcal{B}(\lambda, \ell))$ is a crystal base of $L_q(\mathfrak{d}_{m|n}, \Lambda_{m|n}(\lambda, \ell))$, and the map

$$(5.6) \quad \Psi_{(\lambda, \ell)} : (\mathcal{B}(\lambda, \ell) / \{\pm 1\}) \cup \{0\} \longrightarrow \mathbf{T}_{m|n}(\lambda, \ell) \cup \{0\},$$

given by $\tilde{x}_{i_1} \cdots \tilde{x}_{i_r} \mathbf{v}_{(\lambda, \ell)} \longmapsto \tilde{x}_{i_1} \cdots \tilde{x}_{i_r} \mathbf{H}_{(\lambda, \ell)}^\natural$ for $r \geq 0$, $i_1, \dots, i_r \in I_{m|n}$ and $x = e, f$ is a weight preserving bijection which commutes with \tilde{e}_i and \tilde{f}_i for $i \in I_{m|n}$. Finally, the uniqueness of a crystal base of $L_q(\mathfrak{d}_{m|n}, \Lambda_{m|n}(\lambda, \ell))$ follows from Theorem 5.4(2) and [1, Lemma 2.7(iii) and (iv)]. \square

Corollary 5.8. *Each $U_q(\mathfrak{d}_{m|n})$ -module in $\mathcal{O}_q^{\text{int}}(m|n)$ has a crystal base.*

Corollary 5.9. *Each highest weight $U_q(\mathfrak{d}_{m|n})$ -module in $\mathcal{O}_q^{\text{int}}(m|n)$ is a direct summand of $\mathcal{V}_q^{\otimes M}$ for some $M \geq 1$.*

6. PROOF OF THEOREM 4.3

Let $(\lambda, \ell) \in \mathcal{P}(\mathfrak{d})_{m+n}$ be given. Since $\mathbf{M}_{\mathbb{J}_{m+n} \times 1} = \mathbf{T}_{m+n}^{\text{sp}}$, we may understand $\mathbf{M}_{\mathbb{J}_{m+n} \times \ell}$ as a \mathfrak{gl}_{m+n} -crystal, where $\mathbf{m} \in \mathbf{M}_{\mathbb{J}_{m+n} \times \ell}$ is identified with $\mathbf{m}^{(1)} \otimes \cdots \otimes \mathbf{m}^{(\ell)} \in (\mathbf{T}_{m+n}^{\text{sp}})^{\otimes \ell}$. It is known that $\mathbf{M}_{\mathbb{J}_{m+n} \times \ell}$ is a $(\mathfrak{gl}_{m+n}, \mathfrak{gl}_\ell)$ -bicrystal and the map (3.3) is an isomorphism of bicrystals. Note that \tilde{e}_i, \tilde{f}_i on $\mathbf{T}_{m+n}(\lambda, \ell)$ coincide with those on $\mathbf{M}_{\mathbb{J}_{m+n} \times \ell}$ for $i \in I_{m+n} \setminus \{\bar{m}\}$ since $\mathbf{T}_{m+n}(\lambda, \ell) \subset (\mathbf{T}_{m+n}^{\text{sp}})^{\otimes \ell}$.

Lemma 6.1. *$\mathbf{T}_{m+n}(\lambda, \ell) \cup \{0\}$ is invariant under \tilde{e}_i and \tilde{f}_i for $i \in I_{m+n} \setminus \{\bar{m}\}$, and hence $\mathbf{T}_{m+n}(\lambda, \ell)$ is a \mathfrak{gl}_{m+n} -crystal.*

Proof. Let $\mathbf{T} \in \mathbf{T}_{m+n}(\lambda, \ell)$ be given and let $\mathbf{m} \in \mathbf{M}_{\mathbb{J}_{m+n} \times \ell}$ be the corresponding matrix. If $\tilde{x}_i \mathbf{m} \neq \mathbf{0}$ for some $i \in I_{m+n} \setminus \{\overline{m}\}$ and $x = e$ or f , then we have $Q(\mathbf{m}) = Q(\tilde{x}_i \mathbf{m})$ since (3.3) is an isomorphism of bicrystals, and hence $\tilde{x}_i \mathbf{T} \in \mathbf{T}_{m+n}(\lambda, \ell)$ by (3.16). \square

It remains to show that $\mathbf{T}_{m+n}(\lambda, \ell) \cup \{\mathbf{0}\}$ is invariant under $\tilde{e}_{\overline{m}}, \tilde{f}_{\overline{m}}$. For this, we will show that $\tilde{x}_i(T_2, T_1)$ ($x = e, f$) is also admissible, whenever it is not $\mathbf{0}$, for any admissible pair (T_2, T_1) . We will prove the case when $x = e$ since the proof for $x = f$ is similar.

First, we need the following two lemmas, which can be checked in a straightforward manner using Algorithms 1 and 2 in Section 3.1.

Lemma 6.2. *Let $T \in \mathbf{T}_{m+n}(a)$ be given such that $T' := \tilde{e}_{\overline{m}}T = (\tilde{e}_{\overline{m}}T^{\mathbf{R}}) \otimes T^{\mathbf{L}} \neq \mathbf{0}$.*

Suppose that $\mathbf{r}_T = \mathbf{r}_{T'}$. Then

- (1) ${}^{\mathbf{L}}T' = {}^{\mathbf{L}}T$,
- (2) ${}^{\mathbf{R}}T$ has a domino $\begin{smallmatrix} \overline{m} \\ \overline{m-1} \end{smallmatrix}$, and ${}^{\mathbf{R}}T'$ is obtained from ${}^{\mathbf{R}}T$ by removing it,
- (3) $T'^{\mathbf{L}^*} = T^{\mathbf{L}^*}$, when $\mathbf{r}_T = 1$,
- (4) $T^{\mathbf{R}^*}$ has a domino $\begin{smallmatrix} \overline{m} \\ \overline{m-1} \end{smallmatrix}$, and $T'^{\mathbf{R}^*}$ is obtained from $T^{\mathbf{R}^*}$ by removing it, when $\mathbf{r}_T = 1$.

Suppose that $\mathbf{r}_T \neq \mathbf{r}_{T'}$. Then

- (5) $(\mathbf{r}_T, \mathbf{r}_{T'}) = (1, 0)$ with $\text{ht}(T^{\mathbf{L}}) - a = \text{ht}(T^{\mathbf{R}}) - 2$,
- (6) $T^{\mathbf{L}}$ and ${}^{\mathbf{L}}T$ have exactly one of \overline{m} and $\overline{m-1}$,
- (7) ${}^{\mathbf{L}}T'$ is obtained from ${}^{\mathbf{L}}T$ by removing its top entry,
- (8) ${}^{\mathbf{R}}T$ has a domino $\begin{smallmatrix} \overline{m} \\ \overline{m-1} \end{smallmatrix}$, and ${}^{\mathbf{R}}T'$ is obtained from ${}^{\mathbf{R}}T$ by removing \overline{m} or $\overline{m-1}$, which is different from the top entry of $T^{\mathbf{L}}$,
- (9) $T^{\mathbf{L}^*}$ has a domino $\begin{smallmatrix} \overline{m} \\ \overline{m-1} \end{smallmatrix}$, and $T'^{\mathbf{L}} = T^{\mathbf{L}}$ is obtained from $T^{\mathbf{L}^*}$ by removing \overline{m} or $\overline{m-1}$, which is different from the top entry of $T^{\mathbf{L}}$,
- (10) $T^{\mathbf{R}^*}$ has exactly one of \overline{m} and $\overline{m-1}$ as its entries, and $T'^{\mathbf{R}}$ is obtained from $T^{\mathbf{R}^*}$ by removing it.

\square

Lemma 6.3. *Let $T \in \mathbf{T}_{m+n}(a)$ be given such that $T' := \tilde{e}_{\overline{m}}T = T^{\mathbf{R}} \otimes (\tilde{e}_{\overline{m}}T^{\mathbf{L}}) \neq \mathbf{0}$.*

Suppose that $\mathbf{r}_T = \mathbf{r}_{T'}$. Then

- (1) ${}^{\mathbf{L}}T$ has a domino $\begin{smallmatrix} \overline{m} \\ \overline{m-1} \end{smallmatrix}$, and ${}^{\mathbf{L}}T'$ is obtained from ${}^{\mathbf{L}}T$ by removing it,
- (2) ${}^{\mathbf{R}}T' = {}^{\mathbf{R}}T$,
- (3) $T^{\mathbf{L}^*}$ has a domino $\begin{smallmatrix} \overline{m} \\ \overline{m-1} \end{smallmatrix}$, and $T'^{\mathbf{L}^*}$ is obtained from $T^{\mathbf{L}^*}$ by removing it, when $\mathbf{r}_T = 1$,
- (4) $T'^{\mathbf{R}^*} = T^{\mathbf{R}^*}$, when $\mathbf{r}_T = 1$.

Suppose that $\mathfrak{r}_T \neq \mathfrak{r}_{T'}$. Then

- (5) $(\mathfrak{r}_T, \mathfrak{r}_{T'}) = (0, 1)$ with $\text{ht}(T^{\text{L}}) - a = \text{ht}(T^{\text{R}})$,
- (6) T^{R} and ${}^{\text{L}}T$ have exactly one of \overline{m} or $\overline{m-1}$,
- (7) ${}^{\text{L}}T'$ is obtained from ${}^{\text{L}}T$ by removing its top entry,
- (8) ${}^{\text{R}}T$ has a domino $\boxed{\overline{m}}_{\overline{m-1}}$, and ${}^{\text{R}}T'$ is obtained from ${}^{\text{R}}T$ by removing either \overline{m} or $\overline{m-1}$, which is different from the top entry of T^{R} ,
- (9) $T'^{\text{L}*}$ is obtained from T'^{L} by adding the top entry of T^{R} ,
- (10) $T'^{\text{R}*}$ is obtained from T'^{R} by removing its top entry.

□

Let $T_2 \in \mathbf{T}_{m+n}(a_2)$ and $T_1 \in \mathbf{T}_{m+n}(a_1)$ be given with $a_2 \geq a_1$ and $T_2 \prec T_1$. Suppose that $\tilde{\mathfrak{e}}_{\overline{m}}(T_2, T_1) \neq \mathbf{0}$. For convenience, we put

$$\begin{aligned} a &= a_2 - a_1, \\ (T'_2, T'_1) &= \tilde{\mathfrak{e}}_{\overline{m}}(T_2, T_1), \\ r_i &= \mathfrak{r}_{T_i}, \quad r'_i = \mathfrak{r}_{T'_i}, \\ 2x_i &= \text{ht}(T_i^{\text{L}}) - a_i, \quad 2y_i = \text{ht}(T_i^{\text{R}}), \\ 2x'_i &= \text{ht}(T'^{\text{L}}_i) - a_i, \quad 2y'_i = \text{ht}(T'^{\text{R}}_i), \end{aligned}$$

for $i = 1, 2$. Note that the condition (i) in Definition 3.4(1) is equivalent to $2y_2 \leq 2x_1 + 2r_1r_2$.

Lemma 6.4. *Suppose that $(T'_2, T'_1) = (\tilde{\mathfrak{e}}_{\overline{m}}T_2, T_1)$ with $\tilde{\mathfrak{e}}_{\overline{m}}T_2 = (\tilde{\mathfrak{e}}_{\overline{m}}T_2^{\text{R}}) \otimes T_2^{\text{L}}$. Then $T'_2 \prec T'_1$.*

Proof. We have either $r_2 = r'_2$ or $(r_2, r'_2) = (1, 0)$ by Lemma 6.2(5), and $T_1 = T'_1$.

(1) It is clear that $2y'_2 = 2y_2 - 2 \leq 2x_1 = 2x'_1$ since $x'_i = x_i$ ($i = 1, 2$), $y'_1 = y_1$, and $y'_2 = y_2 - 1$.

(2) If $r_1 = 1$ and $(r_2, r'_2) = (1, 1)$, then by Lemma 6.2(4), we have $T'^{\text{R}*}_2(i) = T^{\text{R}*}_2(i) \leq {}^{\text{L}}T_1(i) = {}^{\text{L}}T'_1(i)$ for $1 \leq i \leq 2y'_2 - 1$. If $r_1 = 1$ and $(r_2, r'_2) = (1, 0)$, then by Lemma 6.2(10), we have $T'^{\text{R}}_2(i) = T^{\text{R}*}_2(i) \leq {}^{\text{L}}T_1(i) = {}^{\text{L}}T'_1(i)$ for $1 \leq i \leq 2y'_2$. If $r_1 = 0$, then it is clear that $T'^{\text{R}}_2(i) = T^{\text{R}}_2(i) \leq {}^{\text{L}}T_1(i) = {}^{\text{L}}T'_1(i)$ for $1 \leq i \leq 2y'_2$.

(3) Suppose that $r_2 = r'_2$. If $r_1 = 1$ and $(r_2, r'_2) = (1, 1)$, then by Lemma 6.2(2) we have ${}^{\text{R}}T'_2(a+i) = {}^{\text{R}}T_2(a+i) \leq T^{\text{L}*}_1(i) = T^{\text{L}*}_1(i)$ for $1 \leq i \leq 2y'_2 + a_1 - 1$. Otherwise, we also have by Lemma 6.2(2) ${}^{\text{R}}T'_2(a+i) = {}^{\text{R}}T_2(a+i) \leq T^{\text{L}}_1(i) = T^{\text{L}}_1(i)$ for $1 \leq i \leq 2y'_2 + a_1 - r'_2$.

Suppose that $(r_2, r'_2) = (1, 0)$. If $r_1 = 0$, then by Lemma 6.2(8) we have ${}^{\text{R}}T'_2(a+i) \leq {}^{\text{R}}T_2(a+i) \leq T^{\text{L}}_1(i) = T^{\text{L}}_1(i)$ for $1 \leq i \leq 2y'_2 + a_1$. So we assume that $r_1 = 1$. Let $u_i = T^{\text{R}}_2(i)$ for $1 \leq i \leq 2y_2$, and let $u'_i = {}^{\text{R}}T'_2(i)$ for $1 \leq i \leq 2y_2 + a_2 - 1 =: N$.

Note that

$$(6.1) \quad u'_r \leq u_{r-a_2+1}$$

for $a_2 \leq r \leq N-2$ by definition of ${}^R T_2$, where $u'_{N-1} = \overline{m-1}$ and $u'_N = \overline{m}$ by Lemma 6.2(8).

Let $v_i = T_1^L(i)$ for $1 \leq i \leq 2x_1 + a_1$ and $v_i^* = T_1^{L^*}(i)$ for $1 \leq i \leq 2x_1 + a_1 + 1$. Then there exists p such that

- $v_i^* = v_i$ for $1 \leq i \leq p$,
- $v_{p+1}^* = w$ for some entry w in T_1^R ,
- $v_i^* = v_{i-1}$ for $p+2 \leq i \leq 2x_1 + a_1 + 1$,

where $p+1 \geq a_1$ since $\sigma(T_1^L, T_1^R) = (a_1 - 1, 2y_1 - 2x_1 - 1)$. Let $v'_i = {}^L T_1(i)$ for $1 \leq i \leq 2x_1 + 1$. Then we have

- $v'_1 = v_{i_1}, \dots, v'_{p-a_1+1} = v_{i_{p-a_1+1}}$ for some $1 \leq i_1, \dots, i_{p-a_1+1} \leq p$,
- $v'_{p-a_1+k} = v_{p+k-1}$ for $k \geq 2$.

Since $T_2 \prec T_1$, we have by Definition 3.4(1)(ii)

$$(6.2) \quad u_{r-a_1+1} \leq v'_{r-a_1+1} = v_r$$

for $p+1 \leq r \leq 2y_2 + a_1 - 3$, and $T_2^{R^*}(2y_2 - 1) \leq v_{2y_2+a_1-2}$, where $T_2^{R^*}(2y_2 - 1) = T_2^L(2x_2)$. Combining (6.1) and (6.2), we get

$$(6.3) \quad u'_r \leq v_{r-a}$$

for $p+a+1 \leq r \leq N-2$. Since ${}^R T'_2(i) = {}^R T_2(i)$ for $1 \leq i \leq N$ by Lemma 6.2(8), we have by (6.3)

$$(6.4) \quad {}^R T'_2(i+a) = u'_{i+a} \leq v_i = T_1^L(i)$$

for $p+1 \leq i \leq N-a-2 = 2y'_2 + a_1 - 1$. Note that $T_1^{L^*}(i) = T_1^L(i)$ for $1 \leq i \leq p$ and hence

$$(6.5) \quad {}^R T'_2(i+a) = {}^R T_2(i+a) \leq T_1^L(i)$$

for $1 \leq i \leq p$. Also, we have ${}^R T'_2(N-1) = T_1^L(2x_1) = T_2^{R^*}(2y_2 - 1) \leq v_{2y_2+a_1-2} = T_1^L(2y_2 + a_1 - 2) = T_1^L(N-a-1)$. Hence by (6.4) and (6.5) we conclude that ${}^R T'_2(i+a) \leq T_1^L(i) = T_1^L(i)$ for $1 \leq i \leq 2y'_2 + a_1$.

Therefore, we have $T'_2 = (\mathbf{e}_{\overline{m}} T_2) \prec T_1 = T'_1$ by (1), (2) and (3). \square

Lemma 6.5. *Suppose that $(T'_2, T'_1) = (\tilde{\mathbf{e}}_{\overline{m}} T_2, T_1)$ with $\tilde{\mathbf{e}}_{\overline{m}} T_2 = T_2^R \otimes (\tilde{\mathbf{e}}_{\overline{m}} T_2^L)$. Then $T'_2 \prec T'_1$.*

Proof. We have either $r_2 = r'_2$ or $(r_2, r'_2) = (0, 1)$ by Lemma 6.3(5), and $T_1 = T'_1$.

(1) It is clear that $2y'_2 \leq 2x'_1 + 2r'_1 r'_2$ since $y'_i = y_i$ ($i = 1, 2$), $x'_1 = x_1$, $x'_2 = x_2 - 1$, and $r'_1 = r_1$.

(2) If $r_1 = 1$ and $(r_2, r'_2) = (1, 1)$, then $T_2^{\mathbf{R}*} = T_2^{\mathbf{R}*}$ by Lemma 6.3(4) and hence $T_2^{\mathbf{R}*}(i) = T_2^{\mathbf{R}*}(i) \leq {}^{\mathbf{L}}T_1(i) = {}^{\mathbf{L}}T'_1(i)$ for $1 \leq i \leq 2y'_2 - 1$. If $r_1 = 1$ and $(r_2, r'_2) = (0, 1)$, then by Lemma 6.3(10), $T_2^{\mathbf{R}*}(i) = T_2^{\mathbf{R}}(i) \leq {}^{\mathbf{L}}T_1(i) = {}^{\mathbf{L}}T'_1(i)$ for $1 \leq i \leq 2y'_2 - 1$. If $r_1 = 0$, then it is clear that $T_2^{\mathbf{R}}(i) = T_2^{\mathbf{R}}(i) \leq {}^{\mathbf{L}}T_1(i) = {}^{\mathbf{L}}T'_1(i)$ for $1 \leq i \leq 2y'_2$.

(3) Suppose that $r_2 = r'_2$. Then we have ${}^{\mathbf{R}}T_2 = {}^{\mathbf{R}}T'_2$ by Lemma 6.3(2) and hence ${}^{\mathbf{R}}T'_2(a+i) = {}^{\mathbf{R}}T_2(a+i) \leq T_1^{\mathbf{L}}(i) = T_1^{\mathbf{L}}(i)$ or ${}^{\mathbf{R}}T'_2(a+i) \leq T_1^{\mathbf{L}*}(i) = T_1^{\mathbf{L}*}(i)$ for $1 \leq i \leq 2y'_2 + a_1 - r'_2$.

Suppose that $(r_2, r'_2) = (0, 1)$. By Lemma 6.3(8), we have ${}^{\mathbf{R}}T'_2(a+i) \leq {}^{\mathbf{R}}T_2(a+i) \leq T_1^{\mathbf{L}}(i) = T_1^{\mathbf{L}}(i)$ or ${}^{\mathbf{R}}T'_2(a+i) \leq T_1^{\mathbf{L}}(i) \leq T_1^{\mathbf{L}*}(i)$ for $1 \leq i \leq 2y'_2 + a_1$.

Therefore, we have $T'_2 = (\tilde{\mathbf{e}}_m T_2) \prec T_1 = T'_1$ by (1), (2) and (3). \square

Lemma 6.6. Suppose that $(T'_2, T'_1) = (T_2, \tilde{\mathbf{e}}_m T_1)$ with $\tilde{\mathbf{e}}_m T_1 = (\tilde{\mathbf{e}}_m T_1^{\mathbf{R}}) \otimes T_1^{\mathbf{L}}$. Then $T'_2 \prec T'_1$.

Proof. We have either $r_1 = r'_1$ or $(r_1, r'_1) = (1, 0)$ by Lemma 6.2(5), and $T_2 = T'_2$.

(1) Note that $x'_i = x_i$ ($i = 1, 2$), $y'_1 = y_1 - 1$, $y'_2 = y_2$, and $r'_2 = r_2$. If $r_1 = r'_1$, then it is clear that $2y'_2 \leq 2x'_1 + 2r'_1 r'_2$.

Suppose that $(r_1, r'_1) = (1, 0)$. If $y_2 \leq x_1$, then we have $2y'_2 \leq 2x'_1 = 2x_1 + 2r'_1 r'_2$. Now, we claim that we have a contradiction when $y_2 > x_1$ (or $y_2 = x_1 + 1$), that is, $2y_2 = 2x_2 + 2r_1 r_2$ with $r_1 = r_2 = 1$. Since $(r_1, r'_1) = (1, 0)$, we have $y_1 = x_1 + 1$. By Lemma 6.2(6) and Definition 3.4(1)(ii), the top entry of $T_2^{\mathbf{R}*}$ is no greater than $\overline{m-1}$. On the other hand, by Lemma 6.2(9) and Definition 3.4(1)(iii), ${}^{\mathbf{R}}T_2$ has a domino $\begin{smallmatrix} \overline{m} \\ \overline{m-1} \end{smallmatrix}$, which also implies that the top entry of $T_2^{\mathbf{R}}$ is \overline{m} . If $x_2 + 1 < y_2$, then $T_2^{\mathbf{R}}$ has a domino $\begin{smallmatrix} \overline{m} \\ \overline{m-1} \end{smallmatrix}$, and $\tilde{\mathbf{e}}_m(T_2, T_1) = (\tilde{\mathbf{e}}_m T_2, T_1)$, which is a contradiction. Next, assume that $x_2 + 1 = y_2$. Put $y = 2y_2$. Consider $T_2^{\mathbf{R}*}(y-1)$, the top entry of $T_2^{\mathbf{R}*}$. If $T_2^{\mathbf{R}*}(y-1) = T_2^{\mathbf{R}}(y) = \overline{m}$, then $T_2^{\mathbf{L}}(y-2) = T_2^{\mathbf{R}}(y) = \overline{m}$. But this implies that the first two top entries of $T_2^{\mathbf{R}}$ are equal to those of ${}^{\mathbf{R}}T_2$, which are \overline{m} and $\overline{m-1}$. So we have a contradiction $\tilde{\mathbf{e}}_m(T_2, T_1) = (\tilde{\mathbf{e}}_m T_2, T_1)$. If $T_2^{\mathbf{R}*}(y-1) = T_2^{\mathbf{R}}(y-1)$, then $T_2^{\mathbf{R}}(y) = \overline{m}$ and $T_2^{\mathbf{R}}(y-1) = \overline{m-1}$ since $T_2^{\mathbf{R}*}(y-1) \leq \overline{m-1}$, which also yields a contradiction $\tilde{\mathbf{e}}_m(T_2, T_1) = (\tilde{\mathbf{e}}_m T_2, T_1)$. This proves our claim.

(2) Suppose that $r_1 = r'_1$. Then by Lemma 6.2(1) we have ${}^{\mathbf{L}}T_1 = {}^{\mathbf{L}}T'_1$ and hence $T_2^{\mathbf{R}}(i) = T_2^{\mathbf{R}}(i) \leq {}^{\mathbf{L}}T_1(i) = {}^{\mathbf{L}}T'_1(i)$ or $T_2^{\mathbf{R}*}(i) = T_2^{\mathbf{R}*}(i) \leq {}^{\mathbf{L}}T_1(i) = {}^{\mathbf{L}}T'_1(i)$ for $1 \leq i \leq 2y'_2 - r'_2$.

Suppose that $(r_1, r'_1) = (1, 0)$, where we have $y_2 \leq x_1$ by (1). If $r_2 = 0$, then by Lemma 6.2(7) we have $T_2'^R(i) = T_2^R(i) \leq {}^L T_1(i) = {}^L T_1'(i)$ for $1 \leq i \leq 2y_2'$. So we assume that $r_2 = 1$.

Let $u_i = T_2^R(i)$ for $1 \leq i \leq 2y_2$ and $u_i^* = T_2^{R*}(i)$ for $1 \leq i \leq 2y_2 - 1$. There exists $p \geq 1$ such that

$$\begin{aligned} \cdot u_i^* &= u_i \text{ for } 1 \leq i \leq p, \\ \cdot u_i^* &= u_{i+1} \text{ for } p+1 \leq i \leq 2y_2 - 1 \end{aligned}$$

Let $u_i' = {}^R T_2(i)$ for $1 \leq i \leq 2y_2 + a_2 - 1$. Then we have

$$(6.6) \quad u_{p+a_2+i-1}' = u_{p+i}$$

for $1 \leq i \leq 2y_2 - p$. Let $v_i = T_1^L(i)$ for $1 \leq i \leq 2x_1 + a_1$ and $v_i^* = T_1^{L*}(i)$ for $1 \leq i \leq 2x_1 + a_1 - 1$. Then we see that for $1 \leq i \leq 2x_1 + a_1 - 1$,

$$(6.7) \quad v_i^* = v_i$$

while $v_{a_1+2x_1}^* = \overline{m-1}$, $v_{a_1+2x_1+1}^* = \overline{m}$, and $v_{a_1+2x_1}$ is either \overline{m} or $\overline{m-1}$ by Lemma 6.2(9). Since $T_2 \prec T_1$, we have by Definition 3.4(1)(iii) $u_{a+i}' \leq v_i^*$ for $1 \leq i \leq 2y_2 + a_1 - 1$. Since $y_2 \leq x_1$, we have by (6.7)

$$(6.8) \quad u_{a+i}' \leq v_i$$

for $1 \leq i \leq 2y_2 + a_1 - 1$. On the other hand, let $v_i' = {}^L T_1(i)$ for $1 \leq i \leq 2x_1 + 1$. Since $r_1 = 1$, we have $v_{a_1+i-1}' \leq v_i'$ for $1 \leq i \leq 2x_1 - 1$.

Now consider $T_2'^R = T_2^R$ and ${}^L T_1'$. By Lemma 6.2(7) we have ${}^L T_1(i) = {}^L T_1'(i)$ for $1 \leq i \leq 2x_1$. Since $u_i^* = u_i$ for $1 \leq i \leq p$, we have

$$(6.9) \quad T_2'^R(i) \leq {}^L T_1'(i)$$

for $1 \leq i \leq p$. By (6.6) and (6.8), we have

$$(6.10) \quad T_2'^R(p+i) = u_{p+i} \leq v_{a_1+p+i-1} \leq v_{p+i}' = {}^L T_1'(p+i)$$

for $1 \leq i \leq 2y_2 - p$, which implies that $T_2'^R(i) \leq {}^L T_1'(i)$ for $p+1 \leq i \leq 2y_2$. Therefore, $T_2'^R(i) \leq {}^L T_1'(i)$ for $1 \leq i \leq 2y_2$ by (6.9) and (6.10).

(3) If $(r_1, r'_1) = (1, 1)$ and $r_2 = 1$, then by Lemma 6.2(3) we have $T_1^{L*} = T_1'^{L*}$ and hence ${}^R T_2'(a+i) \leq T_1^{L*}(i) = T_1'^{L*}(i)$ for $1 \leq i \leq 2y_2' + a_1 - 1$. If $(r_1, r'_1) = (1, 0)$ and $r_2 = 1$, then by Lemma 6.2(9) we have ${}^R T_2'(a+i) \leq T_1^{L*}(i) = T_1'^L(i)$ for $1 \leq i \leq 2y_2' + a_1 - 1$. If $r_2 = 0$, then it is clear that ${}^R T_2'(a+i) \leq T_1^L(i) = T_1'^L(i)$ for $1 \leq i \leq 2y_2' + a_1$.

Therefore, we have $T_2' = T_2 \prec (\tilde{\mathbf{e}}_{\overline{m}} T_1) = T_1'$ by (1), (2) and (3). \square

Lemma 6.7. Suppose that $(T_2', T_1') = (T_2, \tilde{\mathbf{e}}_{\overline{m}} T_1)$ with $\tilde{\mathbf{e}}_{\overline{m}} T_1 = T_1^R \otimes (\tilde{\mathbf{e}}_{\overline{m}} T_1^L)$. Then $T_2' \prec T_1'$.

Proof. We have either $r_1 = r'_1$ or $(r_1, r'_1) = (0, 1)$ by Lemma 6.3(5), and $T_2 = T'_2$.

(1) Note that $y'_i = y_i$ ($i = 1, 2$), $x'_1 = x_1 - 1$, $x'_2 = x_2$, and $r'_2 = r_2$. If $y_2 \leq x_1 - 1 = x'_1$, then we have $2y'_2 \leq 2x'_1 + 2r'_1 r'_2$. So we assume that $y_2 \geq x_1$, that is, $y_2 = x_1$ or $y_2 = x_1 + 1$.

(i) Suppose that $y_2 = x_1$. If $r_1 = r'_1 = 0$, then by Lemma 6.3(1) and Definition 3.4(1)(ii), $T_2^{\mathbf{R}}$ has a domino $\begin{bmatrix} \overline{m} \\ \overline{m-1} \end{bmatrix}$, which implies that $\tilde{\mathbf{e}}_{\overline{m}}(T_2, T_1) = (\tilde{\mathbf{e}}_{\overline{m}} T_2, T_1)$, a contradiction. So we have $(r_1, r'_1) = (0, 1)$ or $(1, 1)$.

Now, suppose that $r_2 = 0$. If $x_2 < y_2$, then the first two top entries of $T_2^{\mathbf{R}}$ and ${}^{\mathbf{R}}T_2$ are the same, and they are \overline{m} and $\overline{m-1}$ by Definition 3.4(1)(iii). But this implies that $\tilde{\mathbf{e}}_{\overline{m}}(T_2, T_1) = (\tilde{\mathbf{e}}_{\overline{m}} T_2, T_1)$, which is a contradiction. So we have $x_2 = y_2$. Now consider the first two top entries of $T_2^{\mathbf{L}}$ and $T_2^{\mathbf{R}}$. Put $x = 2x_2$, and $w_1 = T_2^{\mathbf{L}}(x)$, $w_2 = T_2^{\mathbf{L}}(x-1)$, $w_3 = T_2^{\mathbf{R}}(x)$, $w_4 = T_2^{\mathbf{R}}(x-1)$. First, we have ${}^{\mathbf{R}}T_2(x) = \overline{m}$, ${}^{\mathbf{R}}T_2(x-1) = \overline{m-1}$ by Definition 3.4(1)(iii), which implies that $w_1 = \overline{m}$. Second, we have $w_3 \leq \overline{m-1}$ since $w_3 \leq {}^{\mathbf{L}}T_1(x) \leq \overline{m-1}$ when $r_1 = 0$, and $w_3 \leq {}^{\mathbf{L}}T_1(x-1) = \overline{m-1}$ when $r_1 = 1$. This implies that $w_2 = \overline{m-1}$, and hence $\tilde{\mathbf{e}}_{\overline{m}}(T_2, T_1) = (\tilde{\mathbf{e}}_{\overline{m}} T_2, T_1)$, which is also a contradiction. So we should have $r_2 = r'_2 = 1$. Hence, it follows that $2y'_2 = 2y_2 = 2x_1 = 2(x_1 - 1) + 2 = 2x'_1 + 2r'_1 r'_2$.

(ii) Suppose that $y_2 = x_1 + 1$ with $r_1 = r_2 = 1$. Since $r_1 = r'_1 = 1$, ${}^{\mathbf{L}}T_1$ has a domino $\begin{bmatrix} \overline{m} \\ \overline{m-1} \end{bmatrix}$ by Lemma 6.3(1). But then $T_2^{\mathbf{R}^*}$ and hence $T_2^{\mathbf{R}}$ has a domino $\begin{bmatrix} \overline{m} \\ \overline{m-1} \end{bmatrix}$, which gives a contradiction $\tilde{\mathbf{e}}_{\overline{m}}(T_2, T_1) = (\tilde{\mathbf{e}}_{\overline{m}} T_2, T_1)$.

Therefore, it follows from (i) and (ii) that $2y'_2 \leq 2x'_1 + 2r'_1 r'_2$, when $y_2 \geq x_1$.

(2) Suppose that $r_1 = r'_1$. Note that $y_2 \leq x_1$ by (1) (ii). If $y_2 < x_1$, then by Lemma 6.3(1), we have $T_2^{\mathbf{R}}(i) = T_2^{\mathbf{R}^*}(i) \leq {}^{\mathbf{L}}T_1(i) = {}^{\mathbf{L}}T'_1(i)$ for $1 \leq i \leq 2y'_2$ or $T_2^{\mathbf{R}^*}(i) = T_2^{\mathbf{R}}(i) \leq {}^{\mathbf{L}}T_1(i) = {}^{\mathbf{L}}T'_1(i)$ for $1 \leq i \leq 2y'_2 - 1$. If $y_2 = x_1$, then $r_1 = r_2 = 1$ by (1) (i), and we also have $T_2^{\mathbf{R}^*}(i) = T_2^{\mathbf{R}}(i) \leq {}^{\mathbf{L}}T'_1(i) = {}^{\mathbf{L}}T_1(i)$ for $1 \leq i \leq 2y'_2 - 1$ by Lemma 6.3(1).

Suppose that $(r_1, r'_1) = (0, 1)$. Then by Lemma 6.3(7), we have $T_2^{\mathbf{R}}(i) = T_2^{\mathbf{R}^*}(i) \leq {}^{\mathbf{L}}T_1(i) = {}^{\mathbf{L}}T'_1(i)$ for $1 \leq i \leq 2y'_2$ when $r_2 = 0$, and $T_2^{\mathbf{R}^*}(i) \leq T_2^{\mathbf{R}}(i) \leq {}^{\mathbf{L}}T_1(i) = {}^{\mathbf{L}}T'_1(i)$ for $1 \leq i \leq 2y'_2 - 1$ when $r_2 = 1$.

(3) Suppose that $r_1 = r'_1$. If $(r_1, r'_1) = (1, 1)$ and $r_2 = 1$, then we have by Lemma 6.3(3) ${}^{\mathbf{R}}T'_2(a+i) = {}^{\mathbf{R}}T_2(a+i) \leq T_1^{\mathbf{L}^*}(i) = T_1^{\mathbf{L}^*}(i)$ for $1 \leq i \leq 2y'_2 + a_1 - 1$. Otherwise it is clear that ${}^{\mathbf{R}}T'_2(a+i) = {}^{\mathbf{R}}T_2(a+i) \leq T_1^{\mathbf{L}}(i) = T_1^{\mathbf{L}}(i)$ for $1 \leq i \leq 2y'_2 + a_1 - r'_2$.

Suppose that $(r_1, r'_1) = (0, 1)$. If $r_2 = 0$, then it is clear that ${}^{\mathbf{R}}T'_2(a+i) = {}^{\mathbf{R}}T_2(a+i) \leq T_1^{\mathbf{L}}(i) = T_1^{\mathbf{L}}(i)$ for $1 \leq i \leq 2y'_2 + a_1$. If $r_2 = 1$ with $y_2 < x_1$, then we have by Lemma 6.3(9) ${}^{\mathbf{R}}T'_2(a+i) = {}^{\mathbf{R}}T_2(a+i) \leq T_1^{\mathbf{L}^*}(i) = T_1^{\mathbf{L}}(i)$ for $1 \leq i \leq 2y'_2 + a_1 - 1$. Suppose that $r_2 = 1$ and $y_2 = x_1$. Put $x = 2x_1$. Since $T_2^{\mathbf{R}}(x) \leq {}^{\mathbf{L}}T_1(x)$ and ${}^{\mathbf{L}}T_1(x) = T_1^{\mathbf{R}}(x)$, we have $T_2^{\mathbf{R}}(x) \leq T_1^{\mathbf{R}}(x)$. Using this fact and

Lemma 6.3(9), we can check that ${}^R T_2(a+x-1) = T_2^R(x) \leq T_1^R(x) = T_1'^{L^*}(x-1)$ and hence ${}^R T_2(a+i) \leq T_1'^{L^*}(i)$ for $1 \leq i \leq 2y'_2 + a_1 - 1$.

Therefore, we have $T'_2 = T_2 \prec (\tilde{\mathbf{e}}_{\overline{m}} T_1) = T'_1$ by (1), (2) and (3). \square

Lemma 6.8. *Suppose that $T_2 \in \mathbf{T}_{m+n}(a_2)$ and $T_1 \in \mathbf{T}_{m+n}^{\text{sp}}$ with $a_2 \geq a_1 := \mathbf{r}_{T_1}$ and $T_2 \prec T_1$. If $\tilde{\mathbf{e}}_{\overline{m}}(T_2, T_1) = (T'_2, T'_1) \neq \mathbf{0}$, then $T'_2 \prec T'_1$.*

Proof. Put $\overline{T}_1 = (T_1, H_{(1^N)})$ for a sufficiently large even integer N . Then $\overline{T}_1 \in \mathbf{T}_{m+n}(a_1)$, where

- $\mathbf{r}_{\overline{T}_1} = \mathbf{r}_{T_1}$,
- $\overline{T}_1^L = T_1$, ${}^L \overline{T}_1 = T_1$,
- $\overline{T}_1^{L^*}$ is obtained by adding the largest entry of $H_{(1^N)}$ at the bottom of T_1 when $\mathbf{r}_{\overline{T}_1} = 1$.

It is not difficult to see that $T_2 \prec T_1$ if and only if $T_2 \prec \overline{T}_1$. Now applying Lemmas 6.4, 6.5, and 6.7 to the pair (T_2, \overline{T}_1) , we conclude that $T'_2 \prec T'_1$. \square

For the admissible pairs (T_2, T_1) in Definition 3.4(2) and (3), we can check without difficulty that if $\tilde{\mathbf{e}}_i(T_2, T_1) = (T'_2, T'_1) \neq \mathbf{0}$ for some $i \in I_{m+n}$, then $T'_2 \prec T'_1$.

Hence by Lemmas 6.1–6.8, we conclude that $\mathbf{T}_{m+n}(\lambda, \ell) \cup \{\mathbf{0}\}$ is invariant under $\tilde{\mathbf{x}}_i$ for $\mathbf{x} = \mathbf{e}, \mathbf{f}$ and $i \in I_{m+n}$, which proves Theorem 4.3 (1).

Lemma 6.9. *$\mathbf{T}_{m+n}(\lambda, \ell)$ is a connected \mathfrak{d}_{m+n} -crystal with highest weight $\Lambda_{m+n}(\lambda, \ell)$.*

Proof. Let $\mathbf{H}_{(\lambda, \ell)} = (T_L, \dots, T_0) \in \mathbf{T}_{m+n}(\lambda, \ell)$ be such that

- T_k is empty for $0 \leq k \leq q_+$ when $\ell - 2\lambda_1 \geq 0$,
- $T_0 = \overline{m}$ and $T_k = \overline{m} \overline{m}$ for $1 \leq k \leq q_-$ when $\ell - 2\lambda_1 \leq 0$,
- $T_{q_{\pm}+k} = H_{(1^{a_k})} \in \mathbf{T}_{m+n}(a_k)$ for $1 \leq k \leq M_{\pm}$, where a_k is as in (3.15).

We claim that any $\mathbf{T} \in \mathbf{T}_{m+n}(\lambda, \ell)$ is connected to $\mathbf{H}_{(\lambda, \ell)}$ under $\tilde{\mathbf{e}}_i$ for $i \in I_{m+n}$, where $\text{wt}(\mathbf{H}_{(\lambda, \ell)}) = \Lambda_{m+n}(\lambda, \ell)$. We use induction on $|\mathbf{T}| = \sum_{k=0}^L |T_k|$. Note that $|\mathbf{H}_{(\lambda, \ell)}| = \sum_{i \geq 1} \lambda_i < |\mathbf{T}|$ for all $\mathbf{T} \in \mathbf{T}_{m+n}(\lambda, \ell) \setminus \{\mathbf{H}_{(\lambda, \ell)}\}$.

Suppose that \mathbf{T} is given. We may assume that $\tilde{\mathbf{e}}_i \mathbf{T} = \mathbf{0}$ for $i \in I_{m+n} \setminus \{\overline{m}\}$ since $|\tilde{\mathbf{e}}_i \mathbf{T}| = |\mathbf{T}|$ whenever $\tilde{\mathbf{e}}_i \mathbf{T} \neq \mathbf{0}$ for $i \in I_{m+n} \setminus \{\overline{m}\}$. So, it is enough to show that $\mathbf{T} = \mathbf{H}_{(\lambda, \ell)}$ or $\tilde{\mathbf{e}}_{\overline{m}} \mathbf{T} \neq \mathbf{0}$, which implies $|\tilde{\mathbf{e}}_{\overline{m}} \mathbf{T}| < |\mathbf{T}|$.

Step 1. Suppose that $\ell - 2\lambda_1 \geq 0$. By Definition 3.4(1), $(T_{q_+}, \dots, T_0) \in SST_{\mathbb{J}_{m+n}}(\alpha)$, where $\alpha = ((q_+ + r_+)^k)/\nu$ for some $k \in 2\mathbb{Z}_{\geq 0}$ and $\nu \in \mathcal{P}$ such that each column of ν is also of even length. Since $\tilde{\mathbf{e}}_i \mathbf{T} = \mathbf{0}$ for $i \in I_{m+n} \setminus \{\overline{m}\}$, (T_{q_+}, \dots, T_0) is a \mathfrak{gl}_{m+n} -highest weight element, and hence each of T_0 , T_i^L , and T_i^R ($1 \leq i \leq q_+$) is a \mathfrak{gl}_{m+n} -highest weight element $H_{(1^d)}$ for some $d \in 2\mathbb{Z}_{\geq 0}$. If T_i is not empty for some $0 \leq i \leq q_+$, then $\tilde{\mathbf{e}}_{\overline{m}}(T_{q_+}, \dots, T_0) \neq \mathbf{0}$, and hence $\tilde{\mathbf{e}}_{\overline{m}} \mathbf{T} \neq \mathbf{0}$ by tensor product rule.

Suppose that $\ell - 2\lambda_1 \leq 0$. By Definition 3.4(2), $(T_{q_-}, \dots, T_0) \in SST_{\mathbb{J}_{m+n}}(\beta)$, where $\beta = ((q_- + r_-)^k)/\nu$ for some $k \in 1 + 2\mathbb{Z}_{\geq 0}$ and $\nu \in \mathcal{P}$ such that each column of ν is of even length. As in (1), each of T_0 , T_i^L , and T_i^R ($1 \leq i \leq q_-$) is $H_{(1^d)}$ for some $d \in 1 + 2\mathbb{Z}_{\geq 0}$. If (T_{q_-}, \dots, T_0) has a column of height greater than 1, then we have $\tilde{e}_{\overline{m}}(T_{q_-}, \dots, T_0) \neq \mathbf{0}$ and hence $\tilde{e}_{\overline{m}}\mathbf{T} \neq \mathbf{0}$. Otherwise $T_i = \overline{m}$ or $\overline{m}\overline{m}$ for $0 \leq i \leq q_-$.

By Step 1, we may assume from now on that T_i is empty for $0 \leq i \leq q_+$ when $\ell - 2\lambda_1 \geq 0$, and $T_0 = \overline{m}$ and $T_i = \overline{m}\overline{m}$ for $1 \leq i \leq q_-$ when $\ell - 2\lambda_1 \leq 0$.

Step 2. Consider $T_{q_{\pm}+1}$. Suppose that $\ell - 2\lambda_1 \geq 0$. Then $T_{q_{\pm}+1}^R$ is empty by Definition 3.4(1) since $T_{q_+} \in \mathbf{T}_{m+n}^{\text{sp}^+}$ is empty. Also, we have $T_{q_{\pm}+1}^L = H_{(1^{a_1})}$ since $(T_{q_{\pm}+1}, \dots, T_0)$ is a \mathfrak{gl}_{m+n} -highest weight element with T_i empty for $0 \leq i \leq q_+$. Hence, $T_{q_{\pm}+1}$ is a \mathfrak{d}_{m+n} -highest weight element.

Suppose that $\ell - 2\lambda_1 < 0$. Then $\text{ht}(T_{q_{\pm}+1}^R) \leq 2$ by Definition 3.4(1)(i). Suppose that $\text{ht}(T_{q_{\pm}+1}^R) = 2$ (with $r_{T_{q_{\pm}+1}} = 1$), and let $x = T_{q_{\pm}+1}^R(2)$ and $y = T_{q_{\pm}+1}^R(1)$. By Definition 3.4(1)(ii), we have $x = \overline{m}$. If $y > \overline{m} - 1$, then $\tilde{e}_i(T_{q_{\pm}+1}, \dots, T_0) \neq \mathbf{0}$ for some $i \in I_{m+n} \setminus \{\overline{m}\}$, which is a contradiction. So we have $y = \overline{m} - 1$. Then $\tilde{e}_{\overline{m}}T_{q_{\pm}} \neq \mathbf{0}$ and hence $\tilde{e}_{\overline{m}}\mathbf{T} \neq \mathbf{0}$ since $T_i = \overline{m}$ or $\overline{m}\overline{m}$ for $0 \leq i \leq q_-$. If $T_{q_{\pm}+1}^R$ is empty, then by similar arguments as in (2), we have $T_{q_{\pm}+1}^L = H_{(1^{a_1})}$.

Step 3. By Step 2 we may assume that $T_{q_{\pm}+1}$ is a \mathfrak{d}_{m+n} -highest weight element. Suppose that there exists $k \geq 1$ such that $T_{q_{\pm}+i}$ is a \mathfrak{d}_{m+n} -highest weight element for $1 \leq i \leq k$. Consider $T_{q_{\pm}+k+1}$. By Definition 3.4(1)(i), we have $T_{q_{\pm}+k+1}^R$ is empty. By Definition 3.4(1)(iii), the first a_k entries of $T_{q_{\pm}+k+1}^L$ from the top are $\overline{m}, \dots, \overline{m} - a_k + 1$. Since \mathbf{T} is a \mathfrak{gl}_{m+n} -highest weight element, we have $T_{q_{\pm}+k+1}^L = H_{(1^{a_{k+1}})}$, and hence $T_{q_{\pm}+k+1}$ is a \mathfrak{d}_{m+n} -highest weight element. Applying this argument inductively, we conclude that $\mathbf{T} = \mathbf{H}_{(\lambda, \ell)}$. \square

This proves Theorem 4.3(2), and completes the proof of the theorem.

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